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BEAMS ON ELASTIC SUPPORTS AND ON CROSS-GIRDERS

by

J. M. KLITCHIEFF

(Professor of Applied Mechanics, The Technical University, Belgrade)

SUMMARY

The beam on elastic supports is often considered as a beam on a continuous elastic medium. This method can be applied only to a large number of equally spaced supports having equal rigidities. The method proposed by the author, giving the solution in the form of rapidly converging series, is not bound by this restriction, and when applied to the problem of beams supported by cross-girders reduces it to a system of linear equations, their number being equal to the number of cross-girders, while the existing approximate method leads to the same number of simultaneous differential equations.

Notation

$m-1$	total number of lateral loads, or of intermediate supports, or of cross-girders
x	distance of cross-section considered, measured from left-hand end of beam
u	deflection of beam at cross-section considered
M, M', M''	bending moments in cross-section considered
$i=1, 2, \ldots, m-1$	number corresponding to a particular lateral load, or intermediate support or cross-girder
ξ_i	distance of cross-section denoted by i from left-hand end of beam
u_i	deflection of beam at cross-section denoted by i
R_i, R'_i, R''_i, R'''_i	lateral load, or pressure exerted by beam on support, or on cross-girder as support
l	length of beam
L	length of cross-girder
I	moment of inertia of cross-section of beam
J	moment of inertia of cross-section of cross-girder

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E	Young's modulus
α_i, α	spring constant of elastic support, or supports if equal
B	l^3/EI
B'	L^3/EJ
Q	given lateral load acting on one beam
n, k, j	integers
A_n	coefficients of trigonometric series representing bending moment in beam
C_n, C'_n, C''_n, C'''_n	coefficients of trigonometric series representing deflection curve of girder
S_n	given by equation (11)
γ, β	coefficients of deflection depending on distribution of load and of cross-girders along beam respectively
$\lvert n-2m \rvert$	absolute value of $n-2m$
θ	$n\pi/(2m)$
K_n, L_n	*see* equations (20)

1. PRELIMINARY

The following relations are used in this paper.

(*i*) If $m-1$ lateral loads act on a beam simply supported at the ends the bending moment at a distance x from the left-hand end can be expressed by the trigonometric series

$$M = -\frac{2}{\pi^2} l \sum_{i=1}^{m-1} R_i \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi\xi_i}{l} \sin\frac{n\pi x}{l}, \quad . \quad . \quad . \quad . \quad (1)$$

in which l denotes the span of the beam and ξ_i the distance of the load R_i from the left-hand end.

(*ii*) The sum

$$\sum_{i=1}^{m-1} \sin\frac{in\pi}{m} = \sin^2\frac{n\pi}{2} \cot\frac{n\pi}{2m}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which n and m are integers, is equal to zero for even values of n and for values of n which are multiples of m. For all other values of n it is equal to $\cot n\pi/(2m)$.

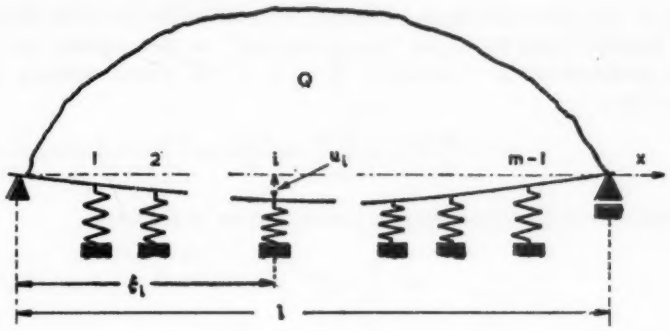

Fig. 1.

(*iii*) The sum

$$2\sum_{i=1}^{m-1}\sin\frac{in\pi}{m}\sin\frac{ik\pi}{m}=\sum_{i=1}^{m-1}\cos\frac{i(n-k)\pi}{m}-\sum_{i=1}^{m-1}\cos\frac{i(n+k)\pi}{m}, \quad . \quad (3)$$

in which n,k and m are integers, is equal to m for $k=n+2jm$, and to $-m$ for $k=-n+2jm$ $(j=1, 2, 3)$. It is equal to zero for all other values of k and, also, for values of n which are multiples of m[1].

(*iv*) By differentiating three times the known series

$$\sum_{j=-\infty}^{\infty}\frac{1}{j\pi+\theta}=\cot\theta$$

it can be proved that

$$\sum_{j=-\infty}^{\infty}\frac{1}{(2jm+n)^4}=\frac{\pi^4}{48m^4}\left(1+\cot^2\frac{n\pi}{2m}\right)\left(1+3\cot^2\frac{n\pi}{2m}\right). \quad . \quad . \quad (4)$$

2. BENDING OF A BEAM ON ELASTIC SUPPORTS

The problem (Fig. 1) of bending of a continuous girder simply supported at the ends on rigid supports and having $m-1$ intermediate elastic supports, is usually discussed by replacing the action of the elastic supports by the action of a continuous elastic medium. The use of this method is limited to the cases of many equally spaced supports of equal rigidities. As shown later these restrictions become superfluous if the expression of the deflection curve of the beam in the form of trigonometric series is used.

The bending moment produced by any given concentrated or distributed load can be written in the form of a series

$$M'=-\frac{\pi^2 EI}{l^2}\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which A_n are given coefficients. Denoting by u_i the deflection of the beam at the support number i and by α_i the "spring-constant" of this support, the bending moment produced by $m-1$ reactions $R_i=\alpha_i u_i$ of the elastic supports becomes (equation (1))

$$M''=\frac{2l}{\pi^2}\sum_{i=1}^{m-1}\alpha_i u_i\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi\xi_i}{l}\sin\frac{k\pi x}{l}. \quad . \quad . \quad . \quad . \quad (6)$$

If the deflection curve of the beam is assumed to be of the form

$$u=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and is introduced into equation (6), this becomes

$$M''=\frac{2l}{\pi^2}\sum_{i=1}^{m-1}\alpha_i\sum_{k=1}^{\infty}C_k\sin\frac{k\pi\xi_i}{l}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi\xi_i}{l}\sin\frac{n\pi x}{l} \quad . \quad . \quad (8)$$

Introducing the expressions (5), (7) and (8) in the differential equation of the deflection curve

$$EI\frac{d^2u}{dx^2}=-(M'+M'')$$

where EI is the rigidity of the beam, a system of linear equations can be obtained of the form

$$n^4C_n+\frac{2}{\pi^4}B\sum_{k=1}^{\infty}C_k\sum_{i=1}^{m-1}\alpha_i\sin\frac{k\pi\xi_i}{l}\sin\frac{n\pi\xi_i}{l}=n^2A_n, \quad . \quad . \quad (9)$$

in which $B=l^3/EI$. This system can be solved easily by successive approximations in every particular case.

In the special case of equally spaced supports of equal rigidities α_i can be replaced by α, ξ_i by li/m. If the values of relation (3) already established are introduced, the system of equations (9) becomes

$$\left.\begin{aligned}
n^4C_n+\frac{m}{\pi^4}\alpha BS_n&=n^2A_n,\\
(n-2m)^4C_{|n-2m|}-\frac{m}{\pi^4}\alpha BS_n&=(n-2m)^2A_{|n-2m|},\\
(n+2m)^4C_{n+2m}+\frac{m}{\pi^4}\alpha BS_n&=(n+2m)^2A_{n+2m},\\
(n-4m)^4C_{|n-4m|}-\frac{m}{\pi^4}\alpha BS_n&=(n-4m)^2A_{|n-4m|},\\
&\cdots\cdots\cdots
\end{aligned}\right\} \quad . \quad . \quad (10)$$

in which the following notation is introduced

$$S_n = C_n + C_{n+2m} + C_{n+4m} + \ldots - C_{|n-2m|} - C_{|n-4m|} - \ldots \quad (11)$$

Eliminating S_n from the first pair of equations (10)

$$C_{|n-2m|} = -\frac{n^4}{(n-2m)^4} C_n + \frac{n^2 A_n}{(n-2m)^4} + \frac{A_{|n-2m|}}{(n-2m)^2},$$

and proceeding in the same way,

$$C_{n+2m} = \frac{n^4}{(n+2m)^4} C_n - \frac{n^2 A_n}{(n+2m)^4} + \frac{A_{n+2m}}{(n+2m)^2},$$

$$C_{|n-4m|} = -\frac{n^4}{(n-4m)^4} C_n + \frac{n^2 A_n}{(n-4m)^4} + \frac{A_{|n-4m|}}{(n-4m)^2},$$

. . . . . .

Thus, equation (11) becomes

$$S_n = (n^4 C_n - n^2 A_n) \sum_{j=-\infty}^{\infty} \frac{1}{(n+2jm)^4} + \frac{A_n}{n^2} + \sum_{j=1}^{\infty} \frac{A_{n+2jm}}{(n+2jm)^2} - \sum_{j=1}^{\infty} \frac{A_{|n-2jm|}}{(n-2jm)^2}.$$

Then, from the first of equations (11), using relation (4), the final equation determining C_n can be obtained

$$\left(C_n - \frac{A_n}{n^2}\right)\left[1 + \frac{\alpha B}{48m^4}\left(1 + \cot^2\frac{n\pi}{2m}\right)\left(1 + 3\cot^2\frac{n\pi}{2m}\right)\right] = \frac{m\alpha}{n^4\pi^4} B\left[\sum_{j=1}^{\infty} \frac{A_{|n-2jm|}}{(n-2jm)^2} - \sum_{j=0}^{\infty} \frac{A_{n+2jm}}{(n+2jm)^2}\right]. \quad (12)$$

3. CROSS-GIRDERS SUPPORTING SEVERAL BEAMS

The method already used can be applied in the case of several equally spaced and equally loaded beams of equal rigidities, supported by a longitudinal girder (Fig. 2). The deflection u_i at the joint number i can be written in the form

$$u_i = B(\gamma Q - \beta R_i), \quad (13)$$

in which Q denotes the load acting on one beam, R_i the pressure exerted by the beam on the girder, and γ and β are coefficients depending on the distribution of

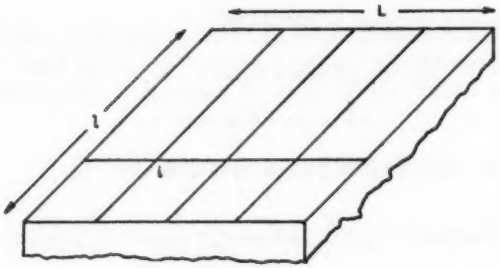

Fig. 2.

the load Q along the beam and the position of the cross-girder. For example, if the load is distributed uniformly and the girder bisects the beams, $\gamma=5/384$ and $\beta=1/48$. From equation (13)

$$R_i=\frac{\gamma}{\beta}Q-\frac{u_i}{B\beta}=\frac{\gamma}{\beta}Q-\frac{1}{B\beta}\sum_{k=1}^{\infty}C_k\sin\frac{k\pi i}{m}. \quad . \quad . \quad . \quad (14)$$

The bending moment produced by the pressures of $m-1$ beams is

$$M=\frac{2}{\pi^2}L\left\{\frac{\gamma}{\beta}Q\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{i=1}^{m-1}\sin\frac{n\pi i}{m}\sin\frac{n\pi x}{L}-\frac{1}{B\beta}\sum_{k=1}^{\infty}C_k\sum_{i=1}^{m-1}\sin\frac{k\pi i}{m}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi i}{m}\sin\frac{n\pi x}{L}\right\}.$$

Introducing this into the differential equation of the deflection curve of the cross-girder and proceeding as in the case of elastic supports,

$$C_n+\frac{2}{\pi^4\beta}\frac{B'}{B}\frac{1}{n^4}\sum_{k=1}^{\infty}C_k\sum_{i=1}^{m-1}\sin\frac{k\pi i}{m}\sin\frac{n\pi i}{m}=\frac{2}{\pi^4}\frac{\gamma}{\beta}B'Q\frac{1}{n^4}\sum_{i=1}^{m-1}\sin\frac{n\pi i}{m},$$

in which $B'=L^3/EJ$, J being the moment of inertia of the cross-section of the girder. Using relations (2), (3) and (4) this equation becomes

$$C_n\left[1+\frac{1}{48\beta m^3}\frac{B'}{B}\left(1+\cot^2\frac{n\pi}{2m}\right)\left(1+3\cot^2\frac{n\pi}{2m}\right)\right]=\frac{2}{\pi^4}\frac{\gamma}{\beta}B'Q\frac{\cot n\pi/2m}{n^4}, \quad (15)$$

R_i being determined by equation (14).

Equations (15) and (14) can be applied also to the case of *two* symmetrically situated girders of equal rigidities. The coefficient β in this case must correspond to the deflection of the beam produced by two equal and symmetrically situated loads.

4. BEAMS SUPPORTED BY SEVERAL GIRDERS

The same method can be applied to the case of several cross-girders, and will be illustrated by the following numerical example.

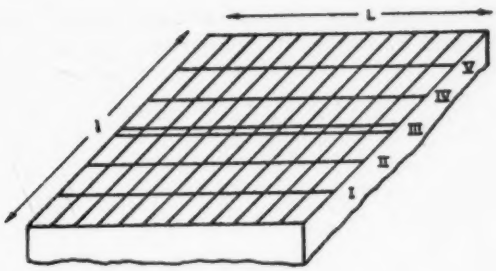

Fig. 3.

Consider (Fig. 3) the case of 11 equally spaced beams bearing uniformly distributed loads Q and supported by 5 equally spaced cross-girders, the moment of inertia of the central girder being twice as large as the moments of inertia of the side-girders. This might correspond, for example, to the bottom structures of a large ship.

Denote by R'_i the pressures exerted by the beam number i on the girders I and V, by R''_i the pressures exerted by it on the girders II and IV, and by $2R'''_i$ the pressure on the central girder. The beam can be considered as being acted on simultaneously by the given load Q, two equal reactions R'_i at distances $l/6$ from its ends, two equal reactions R''_i at distances $l/3$ and two equal reactions R'''_i at distances $l/2$.

The deflections of the beam at its joints will be respectively

$$\left.\begin{aligned} u'_i &= (\gamma' Q - \beta_{11} R'_i - \beta_{12} R''_i - \beta_{13} R'''_i), \\ u''_i &= (\gamma'' Q - \beta_{21} R'_i - \beta_{22} R''_i - \beta_{23} R'''_i), \\ u'''_i &= (\gamma''' Q - \beta_{31} R'_i - \beta_{32} R''_i - \beta_{33} R'''_i). \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

This system of equations corresponds to equation (13).

The numerical values of the coefficients γ can be calculated from the known expression for the deflection of a simply supported beam produced by a uniformly distributed load,

$$\gamma = \frac{1}{24}\frac{x}{l}\left[1 - 2\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right].$$

Replacing x by $l/6$, $2l/6$, $3l/6$, respectively, we obtain

$$\gamma' = \frac{205}{24 \times 6^4}, \quad \gamma'' = \frac{352}{24 \times 6^4}, \quad \gamma''' = \frac{405}{24 \times 6^4}.$$

The values of the coefficients β can be calculated from a similar expression for the deflection produced by two equal concentrated loads at distances $\xi \geqslant x$ from its ends

$$\beta = \frac{1}{6}\frac{x}{l}\left[3\frac{\xi}{l}\frac{l-\xi}{l} - \left(\frac{x}{l}\right)^2\right],$$

giving

$$\beta_{11} = \frac{14}{6^4}, \quad \beta_{12} = \beta_{21} = \frac{23}{6^4}, \quad \beta_{13} = \beta_{31} = \frac{26}{6^4},$$

$$\beta_{22} = \frac{40}{6^4}, \quad \beta_{23} = \beta_{32} = \frac{46}{6^4},$$

$$\beta_{33} = \frac{54}{6^4}.$$

Thus, equation (16) becomes

$$\left.\begin{aligned} 14\,R'_i + 23\,R''_i + 26\,R'''_i &= \frac{205}{24}Q - \frac{u'_i}{B}6^4, \\ 23\,R'_i + 40\,R''_i + 46\,R'''_i &= \frac{352}{24}Q - \frac{u''_i}{B}6^4, \\ 26\,R'_i + 46\,R''_i + 54\,R'''_i &= \frac{405}{24}Q - \frac{u'''_i}{B}6^4. \end{aligned}\right\} \quad . \quad . \quad . \quad (17)$$

Solving this system we have

$$\left.\begin{aligned} R'_i &= \frac{1}{26}\left\{\frac{118}{24}Q - \frac{6^4}{B}(44\,u'_i - 46\,u''_i + 18\,u'''_i)\right\}, \\ R''_i &= \frac{1}{26}\left\{\frac{100}{24}Q - \frac{6^4}{B}(-46\,u'_i + 80\,u''_i - 46\,u'''_i)\right\}, \\ R'''_i &= \frac{1}{26}\left\{\frac{53}{24}Q - \frac{6^4}{B}(18\,u'_i - 46\,u''_i + 31\,u'''_i)\right\}. \end{aligned}\right\} \quad . \quad . \quad (18)$$

corresponding to equation (14); or, writing

$$u' = \sum_{n=1}^{\infty} C'_n \sin\frac{n\pi x}{L}, \quad u'' = \sum_{n=1}^{\infty} C''_n \sin\frac{n\pi x}{L}, \quad u''' = \sum_{n=1}^{\infty} C'''_n \sin\frac{n\pi x}{L},$$

this becomes

$$\left.\begin{aligned} R'_i &= \frac{1}{26}\left\{\frac{118}{24}Q - \frac{6^4}{B}\sum_{k=1}^{\infty}(44\,C'_k - 46\,C''_k + 18\,C'''_k)\sin\frac{k\pi i}{12}\right\}, \\ R''_i &= \frac{1}{26}\left\{\frac{100}{24}Q - \frac{6^4}{B}\sum_{k=1}^{\infty}(-46\,C'_k + 80\,C''_k - 46\,C'''_k)\sin\frac{k\pi i}{12}\right\}, \\ R'''_i &= \frac{1}{26}\left\{\frac{53}{24}Q - \frac{6^4}{B}\sum_{k=1}^{\infty}(18\,C'_k - 46\,C''_k + 31\,C'''_k)\sin\frac{k\pi i}{12}\right\}. \end{aligned}\right\} \quad . \quad (19)$$

Introducing these values in the differential equation of the deflection curve of the first girder, the following is obtained.

$$C'_n=\frac{2B'}{\pi^4 n^4}\sum_{i=1}^{11} R'_i \sin\frac{n\pi i}{12}=\frac{B'}{26\pi^4 n^4}\left\{\frac{2\times 118}{24}Q\cot\frac{n\pi}{24}-\frac{12\times 6^4}{B}(44\,S'_n-46\,S''_n+18\,S'''_n)\right\}$$

in which

$$S'_n=C'_n+C'_{n+2m}+C'_{n+4m}+\ldots-C'_{|n-2m|}-C'_{|n-4m|}-\ldots,$$

$$S''_n=C''_n+C''_{n+2m}+C''_{n+4m}+\ldots-C''_{|n-2m|}-C''_{|n-4m|}-\ldots,$$

$$S'''_n=C'''_n+C'''_{n+2m}+C'''_{n+4m}+\ldots-C'''_{|n-2m|}-C'''_{|n-4m|}-\ldots$$

Hence

$$C'_n+\frac{12\times 6^4}{26\pi^4}\frac{B'}{B}(44\,C'_n-46\,C''_n+18\,C'''_n)\sum_{j=-\infty}^{\infty}\frac{1}{(24j+n)^4}=\frac{2QB'}{26\pi^4 n^4}\frac{118}{24}\cot\frac{n\pi}{24}.$$

Using the notation

$$\left.\begin{aligned}K_n&=\frac{12\times 6^4}{26\pi^4}\frac{B'}{B}\sum_{j=-\infty}^{\infty}\frac{1}{(24j+n)^4}=\frac{6^4}{26\times 48\times 12^3}\frac{B'}{B}\left(1+\cot^2\frac{n\pi}{24}\right)\left(1+3\cot^2\frac{n\pi}{24}\right),\\ L_n&=\frac{2}{24\times 26\pi^4}\frac{1}{n^4}\frac{B'}{B}\cot\frac{n\pi}{24},\end{aligned}\right\}\qquad(20)$$

this can be written in the form

$$(1+44K_n)C'_n-46K_nC''_n+18K_nC'''_n=118L_nQB.\qquad(21)$$

In the same way for the cross-girders II and III

$$\left.\begin{aligned}-46K_nC'_n+(1+80K_n)C''_n-46K_nC'''_n&=100L_nQB,\\ 18K_nC'_n-46K_nC''_n+(1+31K_n)C'''_n&=53L_nQB.\end{aligned}\right\}\qquad(22)$$

This system of three equations corresponds to equation (15). The solutions are

$$C'_n=\frac{118+16744K_n+138500K_n^2}{1+155K_n+2808K_n^2+676K_n^3}L_nQB,$$

$$C''_n=\frac{100+15366K_n+237952K_n^2}{1+155K_n+2808K_n^2+676K_n^3}L_nQB,$$

$$C'''_n=\frac{53+9048K_n+273780K_n^2}{1+155K_n+2808K_n^2+676K_n^3}L_nQB.$$

From equations (18), (21) and (22) final expressions for the reactions R_i can be obtained, and give the solution of the problem

$$R'_i = \frac{Q}{26}\left\{\frac{118}{24} - 6^4 \sum_{n=1,3\ldots} \frac{1546 + 192844\,K_n + 79768\,K_n^2}{1 + 155\,K_n + 2808\,K_n^2 + 676\,K_n^3} L_n \sin\frac{n\pi i}{12}\right\},$$

$$R''_i = \frac{Q}{26}\left\{\frac{100}{24} - 6^4 \sum_{n=1,3\ldots} \frac{134 + 42848\,K_n + 67600\,K_n^2}{1 + 155\,K_n + 2808\,K_n^2 + 676\,K_n^3} L_n \sin\frac{n\pi i}{12}\right\},$$

$$R'''_i = \frac{Q}{26}\left\{\frac{53}{24} - 6^4 \sum_{n=1,3\ldots} \frac{-833 - 124956\,K_n + 35828\,K_n^2}{1 + 155\,K_n + 2808\,K_n^2 + 676\,K_n^3} L_n \sin\frac{n\pi i}{12}\right\}.$$

CONCLUSIONS

The problem discussed contained 18 statically indeterminate reactions. They could be found by solving 18 equations, containing 8 unknown quantities each, or using the approximate method proposed by I. G. Boobnov[2]. Replacing the pressures exerted by the beams on the cross-girders by the pressure of a continuous elastic medium reduces the problem to a system of 3 simultaneous differential equations of the fourth order. This system, in principle, presents no difficulties, but it can hardly be expected that the mathematical work involved can be done by a designer.

The method used in this paper reduces the problem to two systems of 3 linear algebraic equations each. In the general case, if no symmetry exists, the number of equations is equal to the number of cross-girders.

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THE ESTIMATION OF RANGE OF JET-PROPELLED AIRCRAFT

by

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INTRODUCTION

The statement that with modern aircraft and jet propulsion engines it is essential to match the engine and aircraft has become a platitude, without the reasons being quite so well appreciated. It is well known that should the aircraft be operating away from its "design point," involving throttling back of the engine, then the loss of efficiency with either jet or turbine airscrew engines is liable to be serious, and to have a large effect on the range. There is also a firm conviction that very high altitude is necessary in order to give a jet aircraft any range worth mentioning.

It seems worth while to examine in a simple way on what qualities of the engine the range depends. The object of this simplified attack on the problem is two-fold; first of all it enables the factors to be readily appreciated, and second it provides rapid methods of comparing the qualities of two different engine types for range.

An aircraft manufacturer or operator will always be interested in the performance of a particular design of aircraft fitted with particular engines, and this paper is not intended in any way to replace or simplify such essentially detailed calculations. What is intended is a simple method of estimating the maximum range under given conditions of any engine type applied to an aircraft of given characteristics.

A large part of the paper will not be new to many interested in aircraft performance, but a number of the conclusions and "ready reckoner" methods, and the general line of argument, are certainly not generally appreciated. The methods are particularly applicable to jet engines but are easily applied to all types of engine.

Notation

Aircraft data

D aircraft total drag (lb. weight)

σ relative air density

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V	aircraft speed (m.p.h.)
W	aircraft weight (lb.)
v	equivalent air speed for minimum drag
n	$(V/v)\sqrt{\sigma}$, ratio of speed to minimum drag speed at that altitude
R	still-air range (miles)
F	fuel capacity of aircraft (lb. weight)
C	$(E+F)/W$, proportion of aircraft weight allotted to power plant and fuel
a, b	aircraft drag coefficients
N_L	optimum lift/drag ratio
δ	aircraft efficiency factor
A	aspect ratio
C_{D0}	profile drag coefficient
w	wing loading (lb. wt./ft.^{2})$=W/S$
ρ_0	sea level air density$=0.0765$ lb./ft.^{3}
S	wing area (ft.^{2})
C_{Dw}	profile drag coefficient of lifting surfaces
D_B	non-lifting surface drag at unit equivalent air speed
x	$D_B/(SC_{Dw})$, ratio of non-lifting/lifting surface drag

Engine data

E	engine weight (lb.)
X	engine thrust (lb weight)
ν	specific fuel consumption (lb./hr.)/lb. of thrust
k	specific thrust extrapolated to sea level$=X/(\sigma E)$
γ	coefficient of variation of specific fuel consumption with forward speed
ν_p	specific fuel consumption based on power ((lb./hr.)/horse-power)

Suffixes

$_0$	refers to conditions at the commencement of flight
$_{min}$	refers to minimum conditions
$_{max}$	refers to maximum conditions
$_{opt}$	refers to optimum conditions
$_c$	refers to critical conditions in stratosphere

RANGE OF JET-PROPELLED AIRCRAFT

1. GENERAL AIRCRAFT DRAG EQUATION AND CHARACTERISTICS

In the greater part of this paper it is assumed that the effect of Mach number on drag may be neglected. This is justified by the fact that it is generally accepted that it never pays to cruise under conditions where compressibility has increased the drag. Accordingly the results of this paper may be applied, subject to the restriction that predicted speeds are less than the critical values. The effect of this restriction is examined in Section 7.

Under these conditions the aircraft drag is made up of profile and induced drag.

$$D = a\sigma V^2 + b W^2/(\sigma V^2)$$

where D is the total drag, a and b are constants, σ is the relative density, V the aircraft speed, and W is the aircraft weight.

The speed of minimum drag is obtained by differentiation and equating to zero.

$$a\sigma V = b W^2/(\sigma V^3)$$

i.e. minimum drag is obtained when the profile and induced drags are the same. If v is put equal to the equivalent air speed for minimum drag, then

$$a v^2 = b W^2/v^2 = \tfrac{1}{2} D_{\min} = W/(2N_L)$$

where N_L is the optimum lift/drag ratio.

It follows that the aircraft drag may now be written in non-dimensional form

$$2N_L D/W = n^2 + 1/n^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where n is the ratio $\sqrt{\sigma}V/v$ or the ratio of aircraft speed to that of minimum drag at that altitude.

Equation (1) is the most convenient form in which to have the basic drag equation. For all future purposes of this paper, an aircraft is to be specified by three parameters

N_L the optimum lift/drag ratio,

W (or W_0) the aircraft weight which will be a function of time, since fuel is being used up in flight. W_0 is therefore the weight at the start of flight,

v (or v_0) the equivalent air speed for minimum drag which again will vary somewhat in flight, and so v_0 is the value at the start of flight.

In a propulsive system where efficiency is independent of forward speed, *i.e.* an idealised piston or airscrew engine, then since the drag times the range is

the work done, which for a fixed amount of fuel must be constant, it is clear that for a maximum range one must fly at minimum drag speed. So long as this is so the actual altitude of flight does not matter, so far as the aircraft is concerned.

For the greatest endurance it is essential to fly at the minimum power, *i.e.* DV must be a minimum,

or $$\frac{2N_L D}{W} n \frac{v}{\sqrt{\sigma}} = \left(n^3 + \frac{1}{n}\right) \frac{v}{\sqrt{\sigma}} \text{ must be a minimum.}$$

This means that for maximum endurance the aircraft should be flown at the lowest possible altitude, and at a value of n of $1/\sqrt[4]{3}$. Continuous flight at this speed may not be possible and in any case the assumed drag equation may no longer be true so near the stall.

On the other hand, for a propulsive system where specific fuel consumption based on thrust does not vary with speed, *i.e.* an idealised jet engine, the results are somewhat different. If the specific fuel consumption is ν then the time of flight for a given fuel load F is $F/(\nu D)$ and the range is

$$R = FV/(\nu D)$$

or $$\frac{2N_L}{W} \frac{v}{\sqrt{\sigma}} \frac{1}{R} = \frac{\nu}{F} \frac{2N_L}{W} \frac{D}{n} = \frac{\nu}{F}\left(n + \frac{1}{n^3}\right).$$

Hence the maximum range is obtained for a minimum value of $n + 1/n^3$, *i.e.* at $n = \sqrt[4]{3}$. It should be emphasised that this is the maximum range speed for a constant altitude when the pilot is able to adjust the throttle to control the speed. In general it is not advisable to operate a turbine engine at other than maximum continuous cruising throttle setting, in which case the altitude is no longer a free variable. It will be found later that for this case $n = \sqrt[4]{3}$ is no longer the best cruising speed for the aircraft.

For maximum endurance $F/(\nu D)$ should be a maximum, that is, the aircraft flies at the minimum drag speed.

It should be remembered that for a given value of n at which the aircraft is flying, the actual drag is independent of the altitude. The power required, however, increases with altitude.

2. SUMMARY OF CONDITIONS GOVERNING RANGE VARIATION WITH ALTITUDE

It seems worth while now to state in a compact form the reasons why range varies as it does for the various types of power plant.

RANGE OF JET-PROPELLED AIRCRAFT

Fundamentally the aircraft itself does not gain in efficiency from flying at high altitude. It is often stated that jet aircraft fly at high altitudes because the drag is low, but actually at the minimum drag speed the drag is independent of altitude. Thus for the aircraft itself range is uninfluenced by the altitude of flight.

These considerations apply more or less for a piston-engined aeroplane. The only factors which tend to produce an optimum altitude for flight are that for a given aircraft there is usually some variation of airscrew efficiency with forward speed and a variation of the specific output of the engines with altitude. This latter effect is very serious above the full throttle height and generally speaking, it never pays to design for cruising an aircraft above the full throttle height of the engines because this is uneconomical in engine weight and so detracts from the possible fuel load.

For jet engines the picture is very different and apt to be confusing if viewed from the point of view of power, but the basic reason for jet aircraft having an optimum range at very high altitudes is that as the forward speed increases, the thrust remaining constant, the power developed and therefore the efficiency, increases directly as the speed. A limit to the altitude is set, often by Mach number considerations, but also by the fact that as a constant thrust at all altitudes is required, bigger and bigger power plants have to be installed as the altitude is increased. Thus the room left for fuel is reduced and so the range eventually falls. The actual optimum altitude for a particular aircraft is determined by the specific weight of the power plant.

The remaining sections of this paper give analytical expression to these arguments. The aim is to derive the range and the optimum conditions. No account is taken of the equally valuable cruising speed. In many cases it will pay to accept a reduced range at higher speed, in fact this is the selling point of jet propulsion power plants. However it cannot be allowed for in any simple way that has any real bearing on air transport, and in any case the paper gives readily usable expressions for this quantity.

3. BASIC ENGINE PARAMETERS

To propel an aircraft at any given speed and altitude a definite power is required, that is, either a definite size of engine is fitted, or the engine is throttled to suit these conditions. In this paper only maximum range conditions are considered, when it is undesirable to throttle. Thus it is assumed that for a given aircraft the amount of engine that is installed is always adjusted to meet the conditions. The only parameters of an engine that interest us therefore are the specific fuel consumption and the thrust for a given engine weight (specific thrust).

The specific fuel consumption v for a jet engine is usually defined as the fuel consumption (lb. wt.) per hour per lb. of thrust. The fuel consumption and thrust

being measured in the same units however, specific fuel consumption is to this extent independent of the units used. It is felt that some apology is necessary for such a definition since, unlike the specific consumption of a piston engine, it is by no means a measure of efficiency. It is, in fact, dimensionably an energy per second divided by a momentum per second, and thus has the units of velocity. The velocity in question is that of the aircraft. The great use of such a term is that it varies very little with flight conditions, and for many approximate purposes may be considered constant.

The specific thrust X/E is the thrust at any specified conditions divided by the engine weight. It is taken as independent of engine weight E itself, although for a given engine type X/E may easily vary with E. Such variation within the useful range is normally small and is taken into consideration when the actual engine size required is fixed.

Most of the quantities used are usually required in terms of the aircraft weight, that is E/W is the fractional engine weight of the aircraft. For clarity it is preferable to leave the ratio as written rather than to give it some obscure symbol.

In considering the relative performance of various power plants the method adopted is to keep the aircraft characteristics constant. This automatically means keeping the ratio $(E+F)/W_0$ constant for various power plants. It is not inferred that this is automatically the best thing to do, for example if a piston engine were being replaced by a high performance jet engine it might be advantageous to increase the value of $(E+F)/W$, thus increasing the wing loading. Such a change, from the point of view under discussion, would constitute a new aircraft since the minimum drag speed would have been changed.

4. RANGE EQUATIONS WITH ENGINE PARAMETERS

Since $(E+F)/W$ is to be regarded as constant $(=C)$, the equation for range becomes

$$R=\frac{2N_L}{\nu}\frac{v}{\sqrt{\sigma}}\left(C-\frac{E}{W}\right)\frac{n}{n^2+1/n^2} \quad . \quad . \quad . \quad . \quad (2)$$

This equation is reasonably satisfactory for small values of $F/W=C-E/W$, but for larger values, the progressive lightening of the aircraft as the fuel is consumed has an important effect in increasing the range. This will be taken into consideration later. For the moment it is easier to use the simple range formula (2).

It is necessary to introduce into (2) the fact that it is undesirable to operate the engine at any other condition than maximum continuous cruising. To do this it is assumed that

$$X/(\sigma E)=k \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

k is the specific thrust corresponding to sea level conditions. It assumes that the engine thrust is independent of forward speed and directly proportional to σ. The former is certainly a good enough approximation, especially for the range in which we are interested. The second is not sufficiently good to use over a wide range of altitude; it is of course true in the stratosphere, and good enough elsewhere over a small range of altitude. Thus the value of k as given is actually the specific thrust extrapolated to sea level, rather than the actual sea level value.

In the foregoing range formula there are three quantities at one's disposal, σ, E/W and n. For fixed throttle setting there is the second relation (3).

Now
$$k = \frac{X}{\sigma E} = \left\{ \frac{W}{2N_L}(n^2 + 1/n^2) \right\} \Big/ (\sigma E)$$

or
$$\frac{1}{\sigma} = \frac{kE}{W} \frac{2N_L}{n^2 + 1/n^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence
$$R = \frac{(2N_L)^{\frac{1}{2}} v}{\nu} \frac{n}{(n^2 + 1/n^2)^{3/2}} k^{\frac{1}{2}} \left(\frac{E}{W}\right)^{\frac{1}{2}} \left(C - \frac{E}{W}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

All parameters are now independent.

If it is assumed that ν is independent of forward speed then an optimum value of $n = \sqrt[4]{2}$ is found by normal methods. It is generally accepted that for jet-engined aircraft the best cruising speed is 1.18 times the minimum drag speed, provided that the aircraft is powered suitably so that the engines are operating at the correct conditions

An extremely good approximation for the variation of ν with forward speed is: —

$$\nu = \nu_0 + \gamma n ,$$

where γ is a constant depending both on the aircraft and engine. Using this a new optimum value could be found for n, but the value of γ is usually so small that its effect may be neglected. For a piston-engine power plant, the variation of ν is so important that the estimation of range would have to be carried out graphically. In this case it is much better to derive the equations on the basis of specific power fuel consumption as is usually done for such engines.

Using equations (4) and (5) the range for a given power plant type may now be plotted against altitude. The range of all types tends to increase with altitude, reach an optimum value, and then decrease. The reason for such an optimum is that the falling atmospheric pressure and constant thrust require larger and larger engines, which in the end swamps the benefit obtained by flying faster.

Attention will be concentrated first on the optimum altitudes for the different engine types. Differentiating equation (5) with respect to E/W and setting to zero, gives the optimum value of E/W

$$(E/W)_{opt} = C/3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

that is, at the optimum altitude the power plant weight of all jet type engines is the same for a given value of $(E+F)$ and is equal to one third of this value. In fact the engine weight should be one half the fuel weight. It will be seen later that this surprisingly simple result is somewhat modified when allowance is made for using up fuel; nevertheless the result is sufficiently accurate to explode a common belief that jet aircraft will always carry much heavier fuel loads and have correspondingly lighter engines than the equivalent piston-engined aircraft. This is only true when the jet aircraft is designed to operate at much lower altitudes than its optimum value.

For example, if the design altitude were such that the air density were exactly double the optimum value, the value of E/W would be halved, and the fuel weight would be five times the engine weight. Under these conditions the range would be about 89 per cent. of the optimum value. Doubling the air density corresponds to a reduction of altitude from say, 48,000 ft. to 33,000 ft.

The actual optimum altitude is given from equations (4) and (6),

$$\sigma_{opt} = \frac{9}{2\sqrt{2}} \frac{1}{kC N_L} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the optimum range is

$$R_{opt} = 0.42 N_L^{\frac{3}{2}} v C^{\frac{3}{2}} k^{\frac{1}{2}}/\nu \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It will be seen that a criterion for range for a power plant is thus the value of $\sqrt{k}/\nu$, the square root of the specific thrust extrapolated to sea level, divided by the specific fuel consumption under flight conditions. This seems to be a quick "ready reckoner" method for the power plant engineer to assess different engine types.

For the aircraft the optimum range is given by the maximum value of $N_L^{\frac{3}{2}} v$. This point will be referred to later.

5. EFFECT OF REDUCTION IN AIRCRAFT WEIGHT DUE TO CONSUMPTION OF FUEL

For a long-range aircraft in which the fuel might be as high as 40 per cent. of the aircraft weight, the change of weight as the aircraft flies is no longer

negligible. It is easily allowed for by writing equation (2) in differential form

$$dR = \frac{2N_L}{\nu} \frac{v}{\sqrt{\sigma}} \frac{n}{n^2 + 1/n^2} \left(\frac{-dW}{W}\right).$$

Eliminating σ by means of equation (4),

$$dR = \frac{(2N_L)^{3/2} v}{\nu} \frac{n}{(n^2 + 1/n^2)^{3/2}} k^{\frac{1}{2}} \left(\frac{E}{W}\right)^{\frac{1}{2}} \left(\frac{-dW}{W}\right).$$

Now v is still a variable with aircraft weight, actually $v \propto \sqrt{W}$, so that

$$dR = (2N_L)^{3/2} v_0 \frac{n}{(n^2 + 1/n^2)^{3/2}} \frac{k^{\frac{1}{2}}}{\nu} \left(\frac{E}{W_0}\right)^{\frac{1}{2}} \left(\frac{-dW}{W}\right),$$

and integrating between the limits of full and empty fuel load gives

$$\mathrm{R} = -(2N_L)^{3/2} v_0 \frac{n}{(n^2 + 1/n^2)^{3/2}} \frac{k^{\frac{1}{2}}}{\nu} \left(\frac{E}{W_0}\right)^{\frac{1}{2}} \log\left(1 - C + \frac{E}{W_0}\right). \quad . \quad (9)$$

This is the formula equivalent to (5) taking full advantage of progressive weight reduction. It assumes that the aircraft is climbing at such a rate as to keep at the correct altitude, and in fact the cruising speed remains constant.

Again an optimum value of the range is obtained for cruising at $n = \sqrt[4]{2}$. Fig. 1 shows plots of equation (9) made in such a manner as to be universal for all aircraft and power plants. This is done by plotting "non-dimensional" range $R/(N_L^{\frac{2}{3}} v_0 \sqrt{k}/\nu)$ against a sort of "non-dimensional" altitude $1/(\sigma k N_L)$. Thus it is only necessary to substitute actual values of N_L and v_0 for a particular aircraft, and $\sqrt{k}/\nu$ for a power plant to obtain the range plot for an actual combination. It will be seen that the range rises to a maximum value and then falls again, as explained earlier.

By differentiating (9) with respect to E/W and setting to zero, the condition for optimum range is:—

$$\left(1 - C + \frac{E}{W_0}\right) \log\left(1 - C + \frac{E}{W_0}\right) = 2\frac{E}{W_0} \quad . \quad . \quad . \quad (10)$$

The solution to this for various values of $C = (E+F)/W_0$ is plotted on Fig. 2, compared with the simple solution $E/W_0 = C/3$ derived earlier. It will be seen that an optimum value of E/W_0 is reached after which it declines rapidly. It is scarcely worth while at this stage to investigate the behaviour of an aircraft of which less than 15 per cent. consists of structure, payload and so forth.

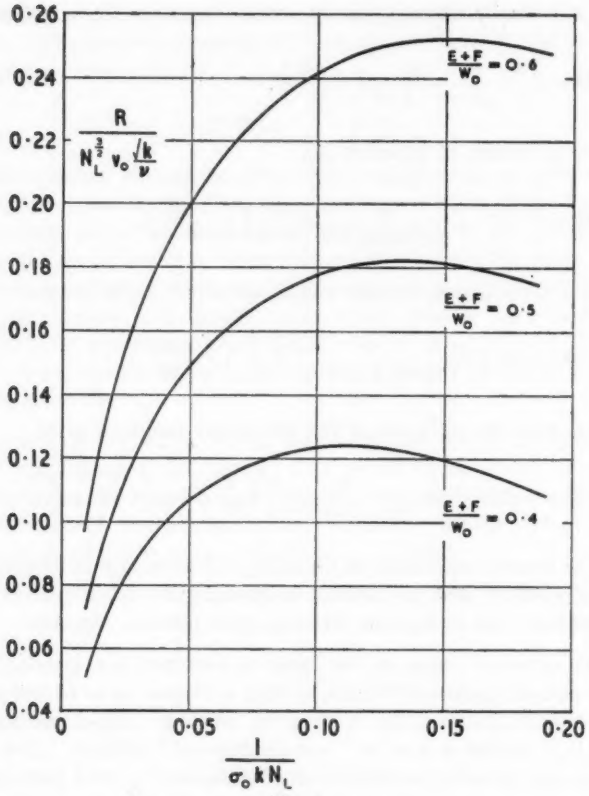

Fig. 1.
Variation of range with altitude.

It will be seen that over a considerable range a fair value of E/W_0 to maintain optimum conditions is

$$E/W_0 = 0.284\ C,$$

in which case the approximate range is given by

$$R_{opt} = 0.33\, N_L^{\frac{3}{2}} v_0 \frac{C^{\frac{3}{2}}}{1 - 0.716\, C} \frac{\sqrt{k}}{\nu} \quad . \quad . \quad . \quad . \quad (11)$$

which is the counterpart of equation (8). Again the determining factor from the power plant point of view is $\sqrt{k}/\nu$ and for the aircraft the value of $N_L^{\frac{3}{2}} v_0$. The very fast rise in range with increasing values of C may also be appreciated from Fig. 3.

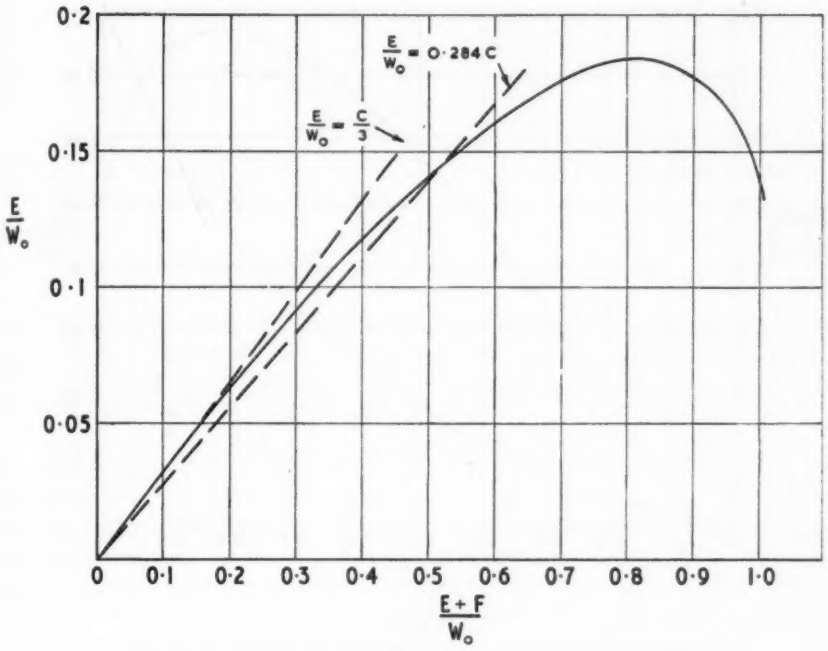

Fig. 2.

Variation of power plant weight with (engine plus fuel) weight for maximum range conditions.

6 OPTIMUM DESIGN OF AIRCRAFT FOR JET PROPULSION

It has been agreed previously that the only noticeable advantage in flying high is that the increased economical cruising speed is more suitable for jet engines. The beneficial effects of the decreased temperature on the engine performance are of a secondary nature. It has been suggested that if an aircraft were designed to have a very high minimum drag speed at low altitude a similar result would be obtained. A high minimum drag equivalent air speed could only be obtained by reducing the wing area and thus increasing markedly the wing loading. Apart from the feasibility of this, however, the previous range formulæ (8) and (11) show that the only criteria of range are the specific thrust, the specific fuel consumption, both of which are not in the least improved by such a step, and the value allotted to $(E+F)/W$, as the factors affected by the engine, and for the aircraft the value of $N_L^{\frac{3}{2}} v$.

Thus the problem of optimum aircraft design for range with jet engines is to

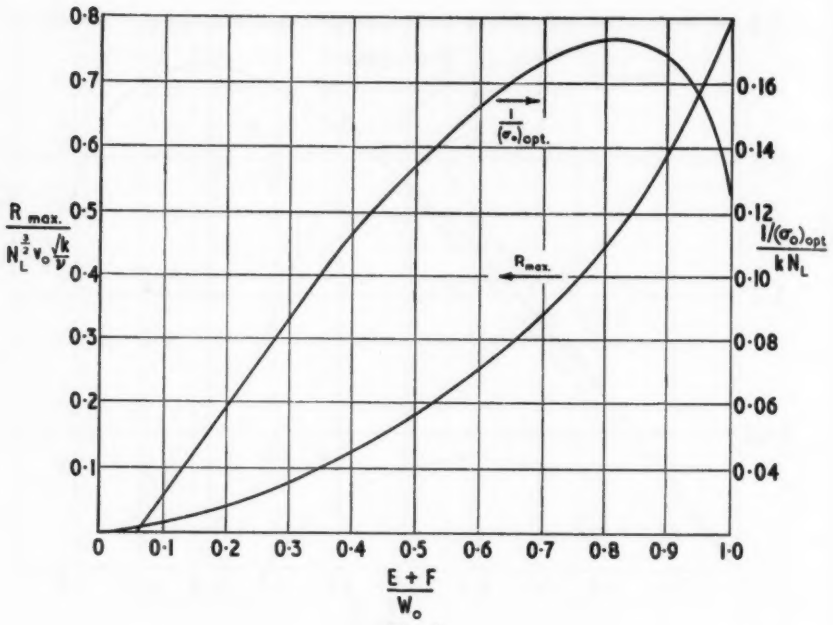

Fig. 3.

Variation of maximum range and optimum altitude with $(E+F)/W_0$.

make the value of $N_L^{\frac{3}{2}} v$ a maximum. These quantities are given in terms of aircraft data by

$$N_L = \sqrt{\left[\frac{\pi}{4}\frac{A}{C_{D0}(1+\delta)}\right]}, \quad v^2 = \sqrt{\left[\frac{w^2(1+\delta)}{A\pi(\frac{1}{2}\rho_0)^2 C_{D0}}\right]}.$$

Thus
$$N_L^{\frac{3}{2}} v = \sqrt{\left[\frac{\pi A}{4(1+\delta)\rho_0}\right]}\sqrt{\left[\frac{w}{C_{D0}^2}\right]}.$$

For the present purpose the aspect ratio is regarded as a constant and variations of wing area are considered. Now the aircraft body will presumably have to remain a fixed size as the wings are reduced, indeed the body would probably have to be made bigger to contain the fuel that would have been contained in the wing, but the body will be regarded as of fixed size and therefore as of fixed drag D_B (assessed at fixed speed).

Hence
$$C_{D0} = C_{DW} + D_B/S,$$

where C_{DW} is the profile drag coefficient of the wing.

Thus the variable term in the expression for $N_L^{\frac{3}{2}} v$ becomes

$$\sqrt{\left[\frac{W}{C_{Dw}^2 S\left(1+\frac{1}{S}\frac{D_B}{C_{Dw}}\right)^2}\right]}.$$

W and C_{Dw} being regarded as constant, this may be differentiated with respect to S and an optimum value is found when

$$\frac{1}{S}\frac{D_B}{C_{Dw}}=1,$$

i.e. the body drag coefficient (expressed on the wing area) should be the same as the wing drag coefficient. Naturally this does not take account of such things as interference drag, but probably this ought to be included as body drag. Obviously the wing drag means the drag of all lifting surfaces.

The condition required seems to be approximately met in most modern aircraft, as the maximum of the expression $\sqrt{x}/(1+x)$ is not very sensitive to values of x. For example at the optimum value of $x=1$ it is 0.5, whereas at $x=2$ and $\frac{1}{2}$ it is still greater than 0.47. On the other hand, to make an aircraft have a very high minimum drag speed at low altitude requires a very high value of $x=D_B/(S C_{Dw})$, approximately $x=29$ to make the minimum drag equivalent air speed 500 m.p.h. Although the jet engine would be operating efficiently, the range would have been reduced to 36 per cent. of the optimum value because of the high body drag.

The fact that jet propulsion exercises an optimum on the value of $N_L^{\frac{3}{2}} v$ is novel and does not occur for a constant power specific engine. The equivalent formula to (5) for such an engine is

$$R=\frac{2N_L}{\nu_p}\frac{1}{n^2+1/n^2}\left(C-\frac{E}{W}\right),$$

where ν_p is the power specific fuel consumption. The conditions for an optimum range therefore are, cruising at minimum drag speed $n=1$, at the minimum value of E/W and therefore at sea level, assuming the specific power does not change much below full throttle height, and for the aircraft a maximum value of N_L, *i.e.* as low a profile drag coefficient as possible.

These conclusions are vitiated in practice, because of the variation of airscrew efficiency, and so forth.

7. EFFECT OF LIMITING MACH NUMBER IN THE STRATOSPHERE

It will be found that for most modern aircraft powered with jet engines, the optimum altitude for range is well above the tropopause and also, the optimum

speed corresponding to this altitude is above the critical Mach number of the aircraft. This places a limit to the possible range below the optimum figure discussed previously. If it is accepted therefore that there is now a constant critical aircraft velocity V_c at which the aircraft must be flown, the conditions for optimum range become somewhat different, and much simpler.

On a simple basis

$$R=(F/v)(V_c/D)$$

so that

$$R=\frac{2N_L}{v}\frac{V_c}{n^2+1/n^2}\left(C-\frac{E}{W}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Now V_c, n and E/W are not independent. In fact

$$\frac{1}{\sigma}=k\frac{E}{W}\frac{2N_L}{n^2+1/n^2}$$

and

$$\frac{1}{\sigma}=\left(\frac{V_c}{nv}\right)^2.$$

Hence

$$R=\frac{2N_L}{v}\frac{V_c}{n^2+1/n^2}\left[C-\frac{(1+1/n^4)}{2N_L k}\left(\frac{V_c}{v}\right)^2\right]$$

$$=\frac{V_c}{v}\left[\frac{2N_L C}{n^2+1/n^2}-\frac{V_c^2}{k n^2 v^2}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

There is no obvious general optimum value of n. It may be found in any specific case and it is clear that in the case of the critical Mach number limiting the altitude for greatest range, the greatest range will be obtained by flying at the critical Mach number at such an altitude that n lies between 1.18 and 1.00, the actual value being determined by the optimum of equation (13).

In practice the assumption of a constant critical Mach number is an over-simplification, since it is likely to be a fast-changing function of the lift coeficient of the wings. As altitude is increased at constant speed, the lift coefficient increases and the limiting Mach number will fall. Such considerations are easy to take into consideration when the exact variation is known.

8. PHYSICAL SIGNIFICANCE OF PREVIOUS RESULTS

It seems worth while to discuss the results summarised in Fig. 1 in terms of real values applied to real aircraft. If a value of N_L, the optimum lift/drag ratio, is taken equal to 18 and an indicated air speed for minimum drag of 180 m.p.h.

typical conditions will be represented. For modern jet power plants figures of $k=3$ are not unreasonable. For an aircraft made up of 50 per cent. engines and fuel, Fig. 1 then shows that the optimum range will be attained at $\sigma=0.137$ or an altitude of above 52,000 ft. at the start of the cruise. Even if $(E+F)/W$ is 40 per cent. then this altitude is still 48,000 ft. and at the end of the flight when the fuel is finished it would have risen to 55,000 ft.

It is evident therefore that to obtain the maximum range from modern jet engines, aircraft will have to fly very high, higher perhaps than may be permissible for civil use. However, as pointed out in an earlier section, the loss in flying lower than the optimum altitude is not too serious. The shape of the curves in Fig. 1 shows that σ may be increased greatly without a large fall-off in range. Thus, if instead of starting at 48,000 ft. in the foregoing example 33,000 ft. is chosen (where σ is exactly double), the range becomes 89 per cent. of the maximum value. At 25,000 ft. the range would be only 80 per cent. of the maximum.

The actual maximum still-air ranges for the cases considered work out at 3,320 miles for the 50 per cent. engine+fuel case and 2,260 miles for the 40 per cent. case.

The fact that the optimum altitude is so high has an important bearing on the increase in range for very light engines that may be developed in the future. Thus if one can fly at the optimum altitude, then the range has been shown to be proportional to $\sqrt{k}/\nu$, *i.e.* if the value of k were increased from 3 to say, 10, then the range would be increased by a factor 1.81. If a limit is set to say, 50,000 ft. as the initial cruising altitude, then this factor will be reduced to 1.3. Thus the returns in increased range from making very light engines will be considerably reduced by limitation of altitude.

This limitation in altitude will be considerable, even for military aircraft, for apart from difficulties of pressurisation, and so on, the aircraft will be restricted by its limiting Mach number. Thus in the example quoted, supposing the limiting Mach number were as high as 0.85, flight would be restricted to 560 m.p.h. which would be attained under normal conditions of cruising, at a little above 50,000 ft. It would then pay to go somewhat higher at the same speed, but the returns would not be great.

Thus from the point of view of increasing the range of more or less conventional aircraft, it would seem that it would be more profitable to concentrate improvements in engines on reducing the specific fuel consumption, while retaining present thrust/weight ratios, rather than to go for much lighter engines.

9. OTHER FACTORS

In this paper conditions for maximum range only have been calculated. It is clear that in practice other considerations may govern the choice of power plant loading, for example take-off or emergency climb. The aircraft may then be effectively over-engined and the engine would have to be operated throttled back. Such considerations come outside the scope of such a general treatment as this and belong to the detailed calculation of performance of a specific aircraft, which it is not the purpose of this paper to cover. It is sufficient to say here that in general, for long-range jet aircraft powered to attain approximately their optimum altitude, there is a large amount of take-off power, and this problem should not therefore be met, except at high altitude aerodromes in hot climates.

Also, in this paper no allowances have been made for fuel used in take-off, climb, descent, stand-off and so on. Here again these factors belong to the specific aircraft calculations. Their qualitative effect is not difficult to see, however; take-off and stand-off as well as reserve fuel represent a constant weight of fuel which cannot be allowed as fuel for range, and should be subtracted from the fuel load before the calculations are made. Climb and descent tend to cancel one another out, but not completely. For long range they do not play a large part in affecting the optimum height for flight, but on shorter range aircraft (low $(E+F)/W$) they may lower the optimum flying height considerably.

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ON NON-STEADY MOTION OF SLENDER BODIES

by

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SUMMARY

Following the original work of Munk, Jones, and Ward for steady flow, a solution is given for unsteady, supersonic flow over very slender bodies of revolution and wings. The results are subject to the restriction $(M^2-1)\delta^2 \log_e [(M^2-1)^{\frac{1}{2}}\delta] << 1$ where δ is the slenderness ratio, M is the Mach number and, in addition, to the usual restrictions imposed by linearisation. As examples, the lifts and pitching moments on flat wings and bodies of revolution executing pitching motion and the damping moment on a rolling wing are calculated.

It is shown that the order of approximation is consistent with the limitations already imposed by linearisation, at least in supersonic flow, where no Kutta condition is required.

Notation

A	wing aspect ratio (*see* equation (28))
C_L	lift coefficient
C_M	pitching moment coefficient
C_l	rolling moment coefficient
D/Dt	differential operator with respect to time
I_n	*see* equation (24)
L	lift (transverse) force
M	Mach number
R	radius of cross section of body of revolution
S	area of wing plan form
S_1	base area for body of revolution
$S(x)$	area of cross section of body of revolution
U	free stream velocity

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a	fraction of body length of axis of pitch aft of nose
b	span at x
c	sonic velocity in free stream
k	reduced frequency (*see* equation (9))
l	body length
m	apparent mass of cross section
p	angular velocity in roll
r	polar co-ordinate
t	time
w	downwash, *i.e.* velocity transverse to free stream
x, y, z	Cartesian co-ordinates
α	amplitude of pitch displacement (*see* equation (19))
β	$(M^2-1)^{\frac{1}{2}}$
γ	semi-apex angle of triangular wing
δ	slenderness ratio (maximum width/length)
θ	polar co-ordinate
ρ	density
ϕ	velocity potential
τ	dimensionless time (Ut/l)
ω	angular frequency

1. INTRODUCTION

The solutions for steady flow past slender bodies of revolution and narrow wings were first given by Munk[1] and Jones[2] respectively. More recently, a critical treatment has been given by Ward[3]. An extension of their work to the problem of unsteady flow is presented here, treating only the lifting forces. While the results of the slender body analysis are independent, in a restricted sense, of Mach number, the supersonic case will be treated primarily since the effect of the wake may then be disregarded.

It is assumed that the body lies in the immediate vicinity of the x axis, its most forward point projecting on the origin and its length being l. The free stream of velocity U is directed along the positive x axis, and the theory is linearised in the usual manner. The velocity potential then satisfies the linearised equation

$$\nabla^2\phi(x, y, z, t)=\frac{1}{c^2}\left(U\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)^2\phi(x, y, z, t), \quad . \quad . \quad (1)$$

where c is the sonic velocity in the undisturbed medium. It is convenient to assume the co-ordinates (x, y, z) to be dimensionless, with l as the unit of length, and to introduce the dimensionless time

$$\tau = Ut/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Expanding the Laplacian and introducing τ, equation (1) reads

$$\phi_{yy} + \phi_{zz} = \beta^2 \phi_{xx} + 2M^2 \phi_{x\tau} + M^2 \phi_{\tau\tau}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$M = U/c, \quad \beta = (M^2 - 1)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The slender body approximation in steady flow assumes the flow to be essentially two-dimensional at every cross section, thereby implicitly neglecting the term $\beta^2 \phi_{xx}$. In the case of a steady flow past a body of revolution, Lighthill[4] has shown that this is the best approximation which is consistently allowed within the framework of the linearised theory. This approximation naturally breaks down for sufficiently large values of β^2, *i.e.* in the hypersonic régime but, since the linearised theory itself is invalid for large M, this does not represent an entirely new limitation. More precisely, if δ is the slenderness ratio, the slender body approximation (in steady flow) requires

$$(\beta\delta)^2 \log_e (\beta\delta) << 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whereas the hypersonic limitation requires $M\delta$ to be small, so that the two restrictions are not exactly equivalent. For a more critical discussion of the slender body approximation in steady flow, the reader is referred to the work of Lighthill[4, 5, 6] and Ward[3].

Turning to the case of unsteady flow, reference is made to a general discussion of the linearising process for the two-dimensional case which has been given recently by Lin, Reissner, and Tsien[7]. They conclude that necessary conditions for the validity of the linearised theory are

$$M\delta << 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$kM\delta << 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where δ is the thickness or amplitude of oscillation, whichever is larger, referred to l, and $1/k$ is a characteristic time, *cf.* equation (9). They also conclude that if M is not large and either $|1 - M|$ or $k >> \delta^{\frac{2}{3}}$ the linearised theory for unsteady flow is valid even in the transonic range. In this connection $\beta\delta$ is the important parameter in the slender body approximation in steady flow, *cf.* equation (5), whereas Ward[3] evidently assumes $\beta = O(1)$ and uses δ alone.

No attempt will be made to discuss here the sufficiency of the foregoing conditions for linearisation of the three-dimensional, unsteady flow problem, other than to point out that, in the case of the body of revolution, there is the additional complication of possible singularities on the axis, as a result of which logarithmic terms are introduced in the order of magnitude approximations. However, a very cursory examination of the analysis of Ref. 7 makes it evident that the restrictions (6) and (7) also will be necessary in the three-dimensional case. Accordingly a two-dimensional cross flow will be assumed and the result examined *a posteriori*, in the light of the restrictions.

2. THE SLENDER BODY SOLUTION

The transverse oscillations, exhibiting the harmonic time dependence $\exp(i\omega t)$, of a pointed body of revolution, described in cylindrical polar co-ordinates by $r=R(x)$, will be considered. The (linearised) boundary condition, to be satisfied at the mean position, is given by

$$\phi_r(x, r, \theta, \tau)\,|_{r=R(x)} = -w(x)\cos\theta\exp(ik\tau) \quad . \quad . \quad . \quad (8)$$

$$k=\omega l/U, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where k is the reduced frequency parameter, and $w(x)$ is the prescribed downwash. There is also the implicit condition that the disturbance vanish at infinity in the proper manner (the Sommerfeld "radiation" condition).

If two-dimensional flow is now assumed at any cross section by neglecting the x derivatives in equation (3),

$$\phi_{rr}+\frac{1}{r}\phi_r+\frac{1}{r^2}\phi_{\theta\theta}=M^2\phi_{\tau\tau}. \quad . \quad . \quad . \quad . \quad (10)$$

A solution to equations (8) and (10), which represents an outgoing disturbance at infinity, is given by

$$\phi(x, r, \theta, \tau)=-\frac{w(x)H_{1(2)}(kMr)\cos\theta\exp(ik\tau)}{kM[H_{1(2)}'(kMr)]_{r=R(x)}} \quad . \quad . \quad . \quad (11)$$

where $H_{1(2)}$ is Hankel's function of the second kind and order one. In the vicinity of the body, where $r\sim R=O(\delta)$, equation (11) reduces to

$$\phi(x, r, \theta, \tau)=w(x)R^2(x)\left(\frac{\cos\theta}{r}\right)\exp(ik\tau)\{1+O[(kM\delta)^2\log_e(kM\delta)]\} \quad (12)$$

Moreover,

$$O\left[\frac{2M^2\phi_{xr}}{\frac{1}{r}\phi_r}\right]=(kM\delta)(M\delta) \quad . \quad . \quad . \quad . \quad (13)$$

$$O\left[\frac{\beta^2\phi_{xx}}{\frac{1}{r}\phi_r}\right]=(M\delta)^2. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It follows that, in view of the restrictions of equations (6), (7), the approximate solution of equation (12) is all that is justified by the linearised theory.

While the foregoing discussion has been limited to a body of revolution, a more general analysis could have been given, following that of Ward. Thus it is found that the linearised flow about any slender body may be approximated as a solution to Laplace's equation provided that, in addition to the restrictions imposed by linearisation, both the maximum angle between any tangent plane to the body and the free stream direction and the rate of change of this angle are $O(\delta)$ and the curvature of any section is $O(1/d)$, where d is the maximum section diameter, at all points where the section is convex outwards (there being no similar restriction at points where the section is concave outwards). The last restriction guarantees small disturbance velocities, relative to U. If it is not satisfied, as in the case of thin flat wings, it can no longer be asserted that equation (12) correctly describes the order of the approximation, but this difficulty is present whenever the linearised theory is applied to thin wings, and the resultant errors in integrated results (such as lifts and moments) generally are acceptable.

In considering bodies of arbitrary cross section, generally it is necessary to consider the interaction of the axial flow, which will introduce a logarithmic term in the potential, and the cross flow, as done by Ward[3]. In the special case of a body of revolution the axial flow has no net effect on the transverse forces, by virtue of symmetry, although naturally it must be included in the calculation of the drag. Thus, the concept of virtual mass may be introduced in the calculation of the transverse forces acting on a body of revolution and, also, on a slender, flat wing, since the axial flow introduces no disturbance in the latter case. (The application of the virtual mass concept to such problems is due to Jones[2].) In the case of a winged body of revolution, on the other hand, the axial flow due to wing-body interference must be considered, and the concept of virtual mass is no longer sufficient for a complete analysis.

Since the solution to equation (3) for unsteady flow is identical with that for steady flow, the difference between the two cases manifests itself entirely in the linearised Bernouilli equation, namely:—

$$\text{pressure}=-\rho\frac{D}{Dt}\phi=-\rho U(\phi_x+ik\phi). \quad . \quad . \quad . \quad . \quad (15)$$

Thus, if $m(x)$ is the virtual mass of a cross section at x, the transverse (lift) force may be calculated directly from Newton's law, namely:—

$$\frac{\partial}{\partial x} L(x) = \frac{D}{Dt}[m(x) w(x) \exp(ik\tau)]$$

$$= U\left(\frac{\partial}{\partial x} + ik\right)[m(x) w(x)] \exp(ik\tau) \quad . \quad . \quad . \quad (16)$$

3. BODY OF REVOLUTION

The potential for this problem has already been obtained, *cf.* equation (12), and equation (16) is applied directly. The virtual mass of a circular cross section is given by

$$m(x) = \rho S(x) = \pi\rho R^2(x) \quad . \quad . \quad . \quad . \quad . \quad (17)$$

whence equation (16) reads

$$\frac{\partial}{\partial x} L(x) = \rho U^2 \left(\frac{\partial}{\partial x} + ik\right)\left[S(x)\frac{w(x)}{U}\right] \exp(ik\tau) \quad . \quad . \quad (18)$$

The case of principal interest is that of a pitching oscillation of amplitude α about $x=a$, for which

$$w(x) = U\alpha[1 + ik(x-a)] \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If the lift and moment (with respect to $x=a$) coefficients are defiined by

$$C_L = [\tfrac{1}{2}\rho U^2 S_1 \exp(ik\tau)]^{-1} \int_0^1 L'(x)\,dx \quad . \quad . \quad . \quad (20)$$

$$C_M = [\tfrac{1}{2}\rho U^2 S_1 l \exp(ik\tau)]^{-1} \int_0^1 (a-x) L'(x)\,dx \quad . \quad . \quad (21)$$

where $S_1 = S(1)$ is the base area, equations (18) and (19) yield

$$C_{L\alpha} = 2[1 + ik(1 - a + I_0) + k^2(aI_0 - I_1)] \quad . \quad . \quad . \quad . \quad (22)$$

$$C_{M\alpha} = 2[-(1 - a - I_0) - ik(1-a)^2 + k^2(I_2 - 2aI_1 + a^2 I_0)] \quad . \quad . \quad (23)$$

$$I_n = \int_0^1 x^n \frac{S(x)}{S_1} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

It is of particular interest that the imaginary part of $C_{M\alpha}$, *i.e.* the damping moment, vanishes for an axis through the base ($a=1$).

4. LOW ASPECT RATIO WINGS

In this section low aspect ratio, pointed wings, are considered, of span $b(x)$, subject to the restriction

$$b'(x) \geqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

In the more general case, where equation (25) is not invoked, it would be necessary to impose the boundary condition

$$\phi(x, y, 0\pm) = \exp[ik(x_1(y) - x)]\,\phi[x_1(y), y, 0\pm] \quad . \quad . \quad (26)$$

at points in the wake aft of the maximum cross section, $x_1(y)$ denoting the trailing edge. Equation (26) ensures continuity of the pressure across the wake, even though the potential itself is discontinuous there. Effectively the imposition of equation (26) spoils the assumption of a two-dimensional cross flow, since the boundary conditions at sections aft of the maximum span are no longer independent.

In the case of steady flow over a flat wing with constant incidence, Jones[2] and Ward[3] have shown that this boundary condition is satisfied by making the cross flow about all cross sections aft of the maximum span identical, so that no lift force arises (*i.e.* $D/Dt = U\partial/\partial x = 0$). Unfortunately this neat state of affairs does not prevail in the case under consideration, both due to the phase factor in equation (26) and the dependence of the local incidence, $w(x)/U$, on x.

Despite the restriction of equation (25), continuity of the pressure also would have to be required at the trailing edge if the flow were subsonic, in satisfaction of the so-called Kutta condition. From the mathematical viewpoint, the Kutta condition, as applied in conventional aerofoil theory, is essentially a uniqueness condition and is required because of the existence of eigensolutions to the boundary value problem. In the case of the slender body solution, however, the solution for the two-dimensional cross flow is already unique for the boundary conditions stated, and no eigensolutions exist. The answer to the apparent paradox lies in the violation of the initial assumptions on which the slender body approximation rests, for if the potential is discontinuous at the trailing edge its x derivatives most certainly cannot be assumed small. Thus, while it has sometimes been stated that the slender body approximation is valid for subsonic, as well as supersonic, flow, this may be asserted with certainty only in those special cases* where the Kutta condition is satisfied automatically. While it may furnish acceptable results in other cases, the order of the error can be established only by a more thorough investigation or by comparison with experiment. Similar objections may be

*Such as the elliptic wing used by Jones[2] as a basis of comparison between the slender body approximation and a known exact result.

levelled at the application of the slender body approximation to the analysis of subsonic flow about a body of revolution having a base of finite area, but in this case no satisfactory theory exists to describe the actual phenomena, which must be heavily influenced by viscous effects. Accordingly this investigation is restricted to supersonic flow over wings with monotonically increasing span (which implies a straight trailing edge, transverse to the flow), so that no section is influenced by events occurring downstream.

Returning to the special category of wings described by equation (25), the approach of equation (16) is used. The virtual mass of a flat plate is identical with that of the circumscribed, circular cylinder, whence

$$m(x)=(\pi/4)\rho b^2(x). \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If the lift and moment coefficients are referred to the plan form area S, and the aspect ratio is introduced

$$A=b_1^2/S, \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where b_1 is the maximum span (in this case, the span at the trailing edge), the results may be taken over from those for the body of revolution by multiplying the latter by $(\pi A/4)$ and replacing $S(x)$ by $b^2(x)$ in equation (24).

The simplest, pointed, low aspect ratio wing, and perhaps the only one of any practical importance, is the triangular wing, for which

$$b(x)/b_1=x, \quad \mathrm{I}_n=(n+3)^{-1}. \quad . \quad . \quad . \quad . \quad (29)$$

In this case

$$C_{\mathrm{L}a}=(\pi A/2)\left[1+ik\left(\frac{4}{3}-a\right)+k^2\left(\frac{a}{3}-\frac{1}{4}\right)\right] \quad . \quad . \quad . \quad . \quad (30)$$

$$C_{\mathrm{M}a}=(\pi A/2)\left[-\left(\frac{2}{3}-a\right)-ik(1-a)^2+k^2\left(\frac{1}{5}-\frac{a}{2}+\frac{a^2}{3}\right)\right]. \quad . \quad (31)$$

5. NON-PLANAR MOTION

Consider now the more general case, where the motion of the wing is not plane. The problem is one in potential flow (because of the slender body approximation) and, in principle, is straightforward.

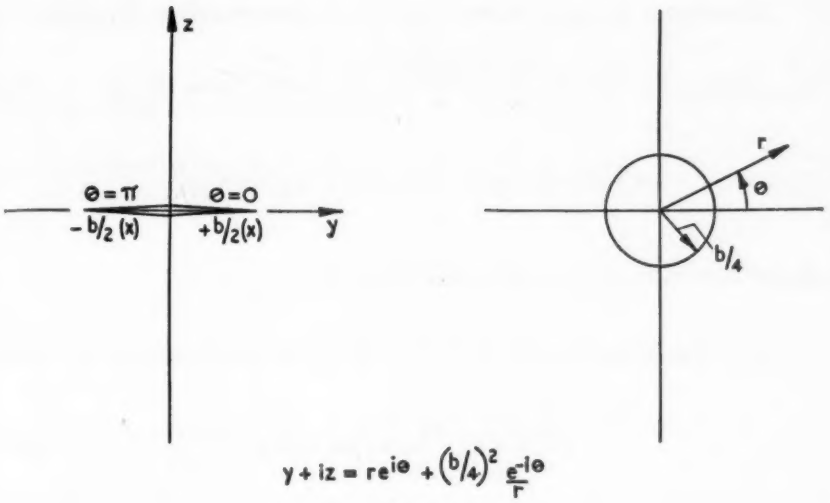

Fig. 1.

Mapping of wing section on circle.

A section of the wing is shown in Fig. 1. The (linearised) boundary conditions are

$$\phi_z(x, y, z)|_{z=0+} = -w(x, y), \quad |y| < \tfrac{1}{2}b(x) \qquad . \quad . \quad . \quad (32)$$

$$\phi(x, y, 0) = 0, \quad |y| > \tfrac{1}{2}b(x) \qquad . \quad . \quad . \quad (33)$$

$$\phi(x, y, z) \to 0, \quad |y|, |z| \to \infty \qquad . \quad . \quad . \quad (34)$$

while ϕ is required to satisfy Laplace's equation

$$\phi_{yy} + \phi_{zz} = 0. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The harmonic time dependence factor has been suppressed.

The desired solution is obtained by mapping the wing ($z=0$, $|y| < b/2$) on the circle $r=(b/4)$ with the aid of the Joukowsky transformation

$$y + iz = re^{i\theta} + \left(\frac{b}{4}\right)^2 \frac{e^{-i\theta}}{r}, \qquad . \quad . \quad . \quad . \quad . \quad (36)$$

as shown in Fig. 1.

Proceeding in the usual manner, a solution to equations (32) to (35) is given by

$$\phi\left(x, \frac{b}{2}\cos\theta, z\right) = \frac{b(x)}{\pi} \sum_{1}^{\infty} \left[\frac{b(x)}{4r}\right]^n \frac{\sin n\theta}{n} \int_0^{\pi} w\left(x, \frac{b}{2}\cos\varphi\right) \sin n\varphi \sin\varphi \, d\varphi \quad . \quad (37a)$$

$$= \frac{b(x)}{2\pi} \int_0^{\pi} w\left(x, \frac{b}{2}\cos\varphi\right) \sin\varphi \log_e \left[\frac{1+(4r/b)^2 - 2(4r/b)\cos(\theta+\varphi)}{1+(4r/b)^2 - 2(4r/b)\cos(\theta-\varphi)}\right] d\varphi. \quad . \quad (37b)$$

On the aerofoil ($r = b/4$), equation (37) reduces to

$$\phi\left(x, \frac{b}{2}\cos\theta, 0+\right) = \frac{b(x)}{\pi} \sum_{1}^{\infty} \frac{\sin n\theta}{n} \int_0^{\pi} w\left(x, \frac{b}{2}\cos\varphi\right) \sin n\varphi \sin\varphi \, d\varphi \quad . \quad (38a)$$

$$= \frac{b(x)}{2\pi} \int_0^{\pi} w\left(x, \frac{b}{2}\cos\varphi\right) \sin\varphi \log_e \left[\frac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}\right] d\varphi. \quad . \quad (38b)$$

For the lift and rolling moment on the wing section at x, the following integrals are required

$$\psi_0(x) = \int_{-\frac{1}{2}b(x)}^{\frac{1}{2}b(x)} \phi(x, y, 0+)\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (39a)$$

$$= \frac{b(x)}{2} \int_0^{\pi} \phi\left(x, \frac{b}{2}\cos\theta, 0+\right) \sin\theta \, d\theta \quad . \quad . \quad . \quad (39b)$$

$$= \frac{b^2(x)}{4} \int_0^{\pi} w\left(x, \frac{b}{2}\cos\theta\right) \sin^2\theta \, d\theta \quad . \quad . \quad . \quad . \quad (39c)$$

$$\psi_1(x) = \int_{-\frac{1}{2}b(x)}^{\frac{1}{2}b(x)} \phi(x, y, 0+)\, y\, dy. \quad . \quad . \quad . \quad . \quad . \quad (40a)$$

$$= \frac{b^2(x)}{4} \int_0^{\pi} \phi\left(x, \frac{b}{2}\cos\theta, 0+\right) \sin\theta\cos\theta \, d\theta \quad . \quad . \quad (40b)$$

$$= \frac{b^3(x)}{32} \int_0^{\pi} w\left(x, \frac{b}{2}\cos\theta\right) \sin 2\theta \sin\theta \, d\theta \quad . \quad . \quad (40c)$$

As an example the damping due to roll (angular velocity p) is considered.

The downwash is given by

$$w(x, y) = py = p[b(x)/2]\cos\theta \quad . \quad . \quad . \quad . \quad (41)$$

and the rolling moment coefficient by

$$C_l = -[\tfrac{1}{2}\rho U^2 S b_1]^{-1} \int_0^1 dx \int_{-\frac{1}{2}b(x)}^{\frac{1}{2}b(x)} [-2p(x, y, 0+)]\, y dy \quad . \quad . \quad (42a)$$

$$= -\frac{4}{Sb_1}\int_0^1 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \tau}\right)\frac{\psi_1(x)}{U} dx \quad . \quad . \quad . \quad . \quad . \quad (42b)$$

Substituting the distribution (41) in (40c) and the latter in equation (42b),

$$C_{l_p} = \frac{\partial C_l}{\partial\left(\frac{pb}{2U}\right)} = -(\pi A/32)\left[1 + ik\int_0^1 \frac{b^4(x)}{b_1^4} dx\right] \quad . \quad . \quad (43)$$

The real and (negative) imaginary parts correspond to the damping and inertia respectively. For a triangular wing of semi-apex angle γ, the aspect ratio $A = 4\tan\gamma$, and equation (43) then gives $-(\pi/8)\tan\gamma$ for the damping moment coefficient. The exact (linearised) result for the damping in roll of a triangular wing in supersonic flow has been calculated by Robinson[8] and, in the limit of small γ, yields

$$C_{l_p} = -(\pi\tan\gamma)/8 + O[(\beta\tan\gamma)^2 \log_e(\beta\tan\gamma)] \, . \quad . \quad . \quad (44)$$

in agreement with the result above for small γ. The comparison reveals that the accuracy of the slender body approximation in predicting integrated force coefficients, relative to the three-dimensional, linearised theory, is correctly indicated by equation (5), despite the violation of the restriction on curvature of convex sections. Similar comparisons also have been made for the results (30), (31) with the limiting forms of the complete longitudinal, stability derivatives[9] (Robinson[8] gives only the quasi-stationary results, neglecting the time derivatives in the Bernoulli equation and the continuity equation).

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THE CORRECTION OF WIND TUNNEL NOZZLES FOR TWO-DIMENSIONAL, SUPERSONIC FLOW

by

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SUMMARY

The paper is concerned with the two-dimensional, steady, irrotational, isentropic flow of a perfect gas in a wind tunnel nozzle which is found to produce a flow in the test rhombus deviating slightly from the desired uniform flow.

The minimum corrections are derived that must be applied to the liners in order to produce a uniform flow in the test rhombus. If the uncorrected nozzle produces a flow of uniform direction, measurement of the pressure on the axis, in the test rhombus, suffices to determine these corrections (Section 5). If not, further pressure measurements are required (Section 6). A simple test is indicated for checking whether the flow stream direction is uniform (Section 6).

The method cannot be used to correct for deviations from a two-dimensional flow.

Notation

x, y	Cartesian co-ordinates in the flow plane, with x measured in the downstream direction of the axis
p	pressure
ρ	density
q	velocity magnitude
μ	Mach angle
a_*	critical speed of sound
t	$\int_{a_*}^{q} (1/q)\cot\mu\, dq$

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θ	stream direction
α	$\theta+t$
β	$\theta-t$
h_α, h_β	length parameters of Mach line pattern (*see* Section 3)
H_α	$(h_\alpha \sin 2\mu)^{-1}$
H_β	$(h_\beta \sin 2\mu)^{-1}$
H_β^0	value of H_β at G (Fig. 1)
M_+, M_-	plus and minus Mach lines (Fig. 2)
S	streamline (Fig. 2)
δ	correction of local inclination of liner
N	$\frac{1}{4}(\gamma+1)\sec^2\mu$
r	length measured on straight minus Mach line from its point of intersection with the plus Mach line BA (Fig. 1)
r_0	value of r on AC (Fig. 1)
x^+	$x-x_A$
$f(\mu)$	$\{[(\gamma-\cos 2\mu)^\gamma (\sin\mu)^{-\gamma-1}]^{1/(\gamma-1)} \sec\mu\}^{\frac{1}{2}}$
γ	ratio of specific heats

The foregoing symbols refer to the theoretical field of flow on which the design of the nozzle is based.

^{m} refers to the measured field of flow in the actual (uncorrected) nozzle.

$'$ denotes differences between measured and theoretical quantities.

* refers to the field of flow in the corrected nozzle.

$''$ denotes differences between quantities referring to the corrected nozzle and those referring to the theoretical field of flow.

1. INTRODUCTION

Several effects in wind tunnel nozzles for two-dimensional, supersonic flow cannot be predicted theoretically, at present, with great accuracy. Sometimes, therefore, the flow obtained in the test rhombus is not as uniform as intended, and it may be desired to make use of the measured distribution of pressure in the working section in order to determine the slight corrections to the liners needed to reduce the deviations from uniformity of the flow below the level of experimental uncertainty. It is not usually necessary that the mean Mach number in the test rhombus should be exactly equal to that aimed at in the original design.

It is assumed that a small correction of the curvature of the downstream part of the liner will not alter the displacement thickness of the boundary layer by a

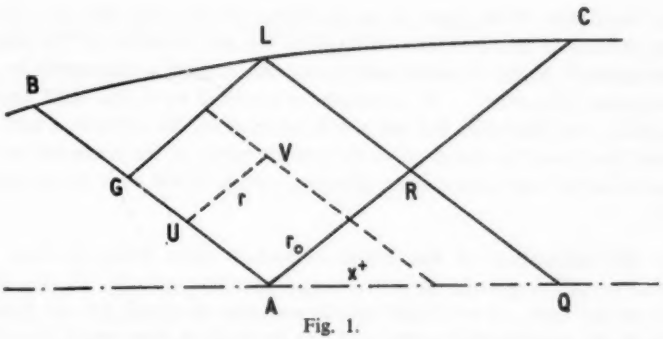

Fig. 1.

significant amount. It is also assumed that the flow in the uncorrected nozzle is indeed two-dimensional. The problem may then be treated by the theory of the two-dimensional, isentropic, irrotational, steady flow of a perfect gas.

A solution of the problem on Linear Theory has been given by Ward[1], a solution on Exact Theory by Harrop and Bright[2]; the latter is not complete, and this may have been the reason why these authors have found it necessary to apply their method of redesign repeatedly in order to obtain the desired uniformity of the flow in practice. Carrière[3] also claims to have obtained a solution.

The problem belongs to a class which has been treated, mainly by Guderley[4], under the name of "Neighbouring Solutions." Two fields of flow are compared in which the respective velocity components, at corresponding points, differ by amounts small enough for the squares and products of such differences to be negligible. In the following a notation developed by one of the authors[5] is employed that yields results more simply and more directly, in the case of two-dimensional flow, than Guderley's notation.

The correction is expressed in terms of the values on the axis of the characteristic co-ordinates of the field of flow in the uncorrected nozzle (equations (13), (14), (20), and (21), (22), (23), which may be found from a minimum of pressure measurements. A procedure for the evaluation of the correction is indicated in Sections 5 and 6.

2. MATHEMATICAL FORMULATION

Let A be the apex of the test rhombus on the axis of the nozzle (Fig. 1; the direction of the flow is from left to right), and let AB and AC be the Mach lines through A, and BC the segment of the liner between them. In a perfect nozzle the region ABC is occupied by a simple wave, since it is adjacent to a region of uniform flow, *i.e.* the test rhombus, and the Mach lines belonging to the same family as AC ("minus" Mach lines) are straight in this region.

The least part of the liner of an imperfect nozzle that must be corrected in order to produce a uniform flow throughout the test rhombus is the segment BC (and the segment placed symmetrically to it with respect to the axis), by virtue of the Uniqueness Theorem[6]. If correction is confined to it, the field upstream of, and including, the Mach line BA will not be affected by the correction, and therefore a uniform flow must be aimed at, in the test rhombus of the corrected nozzle, with that Mach number and that stream direction which obtain at A in the uncorrected nozzle.†

For the calculation of the nozzle correction three fields of flow must be distinguished, differing only little from one another, namely (*i*) the theoretical solution on the basis of which the nozzle was first designed, (*ii*) the field of flow obtained in the uncorrected nozzle, and (*iii*) the field of flow which it is desired to produce in the corrected nozzle. Quantities referring to the second (measured) field will be distinguished in the following by an index "m" from those referring to the first field, and quantities referring to the third field will be distinguished by an asterisk. The difference between the value, at any point, of quantities in the second and the first field will be denoted by a dash, and the difference between values in the third and the first field by two dashes (for example, $\theta^{m}=\theta+\theta'$, $\theta^{*}=\theta+\theta''$). It will be assumed that squares and products of dashed quantities may be neglected.

If x and y denote Cartesian co-ordinates in the flow plane, with x measured in the downstream direction of the axis, and θ denotes the angle the stream direction makes with the direction of x increasing, the slope of the desired liner is

$$dy^{*}/dx=\tan\theta^{*}=\tan(\theta^{m}+\delta),$$

with

$$\delta=\theta^{*}-\theta^{m}=\theta''-\theta', \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and to the first order the liner correction is

$$y^{*}-y^{m}=\int_{x_B}^{x}\delta(1+\tan^{2}\theta)\,dx, \quad . \quad . \quad . \quad . \quad (2)$$

where the integral has to be evaluated along the liner.

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†If it is desired to correct so as to produce a uniform flow with the exact Mach number and stream direction of the original nozzle design, it becomes necessary to extend the correction to a part of the liner upstream of B. To compute such a correction the characteristic equations (18) and (19) of the perturbation problem have to be integrated step by step, by a method analogous to Massau's method[6], in a region upstream of BA.

3. EQUATIONS FOR THE CORRECTION TERMS

The differential equations governing the variation of θ' and θ'' may be derived as follows.

Let
$$t=\int_{a_0}^{q}(1/q)\cot\mu\,dq, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where q denotes the velocity magnitude, μ the Mach angle, and a_0 the critical speed of sound. By virtue of Bernoulli's equation,

$$qdq+(1/\rho)dp=0,$$

where ρ is the density and p the pressure, and we have

$$dp/dt=-\rho q^2\tan\mu. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If
$$\left.\begin{aligned}\alpha&=\theta+t,\\ \beta&=\theta-t,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

then[6]

$$\left.\begin{aligned}dy/dx&=\tan(\theta-\mu) \text{ and } \alpha=\text{constant on any "plus" Mach line,}\\ dy/dx&=\tan(\theta+\mu) \text{ and } \beta=\text{constant on any "minus" Mach line.}\end{aligned}\right\} \quad . \quad (6)$$

As in Ref. 5 the positive direction of a Mach line is defined as that making an acute angle with the stream direction and the elements of length are denoted in the positive directions of the minus and plus Mach lines by $h_\alpha d\alpha$ and $h_\beta d\beta$ respectively. Let

$$H_\alpha=(h_\alpha\sin 2\mu)^{-1} \text{ and } H_\beta=(h_\beta\sin 2\mu)^{-1}. \quad . \quad . \quad . \quad (7)$$

Now compare the Mach line patterns of any two fields of flow differing only by small quantities (the notation already introduced for the first and second field may be used). On the minus Mach lines of the first field $d\beta=0$ by (5), and (Fig. 2)

$$h_\beta{}^{m}d\beta^{m}\sin 2\mu=h_\beta{}^{m}d\beta'\sin 2\mu=h_\alpha d\alpha\,(\theta'+\mu'),$$

to the first order. Since the difference between h_β^m and h_β is small†, it follows that, to the first order,

$$\partial\beta'/\partial\alpha = h_\alpha H_\beta(\theta'+\mu') \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(using (6)). Similarly, on the plus Mach lines (Fig. 2),

$$h_\alpha^m d\alpha^m \sin 2\mu = -h_\beta d\beta(\theta'-\mu'),$$

and hence,

$$\partial\alpha'/\partial\beta = -h_\beta H_\alpha(\theta'-\mu'). \quad . \quad . \quad . \quad . \quad (9)$$

Only one derivative occurs in each of the equations (8) and (9) and they are therefore characteristic equations. Hence, the Mach lines of the first field are also the characteristics in the physical plane of the solution for the dashed quantities‡.

In a region of uniform flow H_α and H_β both vanish identically[5] and hence,

$$\partial\beta'/h_\alpha\partial\alpha = \partial\alpha'/h_\beta\partial\beta = 0. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In a simple wave region where the minus Mach lines are straight H_α vanishes identically[5] and hence,

$$\partial\alpha'/h_\beta\partial\beta = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\partial\beta'/h_\alpha\partial\alpha = H_\beta(\theta'+\mu'). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The variation of quantities with two dashes is governed by analogous equations.

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†This may be seen as follows. By (5),

$$\partial x/\partial\beta = h_\beta\cos(\theta-\mu)$$

$$= \frac{\partial x}{\partial\beta^m}\frac{\partial\beta^m}{\partial\beta} + \frac{\partial x}{\partial\alpha^m}\frac{\partial\alpha^m}{\partial\beta} = h_\beta^m\cos(\theta^m-\mu^m) + \ldots$$

and the difference between $\theta^m-\mu^m$ and $\theta-\mu$ is also small.

‡This fact was first noticed by Guderley[4] and a direct proof has been indicated by Carrière[3].

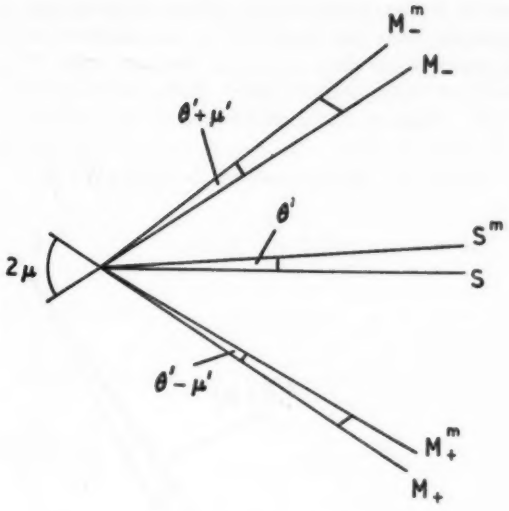

Fig. 2.

4. FORMULÆ AND PROCEDURE

If L is any point on the liner, between B and C (Fig. 1), and G the point on the Mach line BA lying on the same minus Mach line as L, then (2) may be written

$$y^* - y^m = \int_{x_B}^{x} \delta_L (1 + \tan^2 \theta_G)\, dx_L, \qquad . \quad . \quad . \quad . \quad (13)$$

since θ is constant on the straight Mach line GL, and by (1) and (5),

$$\delta_L = \tfrac{1}{2}(\alpha''_L - \alpha'_L + \beta''_L - \beta'_L).$$

This may be expressed in terms of the values of α' on the axis as follows.

By (10) and (11), α' and α'' are constant along the plus Mach line through L and hence, $\alpha'_L = \alpha'_Q$ and $\alpha''_L = \alpha''_Q$, where Q is the point on the axis lying on the plus Mach line through L (Fig. 1). Moreover, α'' is constant on the axis and equal to the value of α' at A, since the correction does not affect the velocity components at that point, so

$$\delta_L = -\tfrac{1}{2}(\alpha'_Q - \alpha'_A + \beta'_L - \beta''_L). \qquad . \quad . \quad . \quad . \quad (14)$$

Let U, U_1 and V, V_1 be respecively the points of intersection of the Mach line BA and of an arbitrary other plus Mach line in the simple-wave region ABC with any two straight minus Mach lines at a small distance apart (Fig. 3). Let O be the common point of these minus Mach lines, and let UU_2 and VV_2 be perpendicular to UV. Then, if $\beta=\beta_V$ on UV and $\beta=\beta_V+d\beta$ on U_1V_1,

$$VV_2=VV_1\sin 2\mu=h_\beta d\beta\sin 2\mu=VO\,d(\theta+\mu),$$

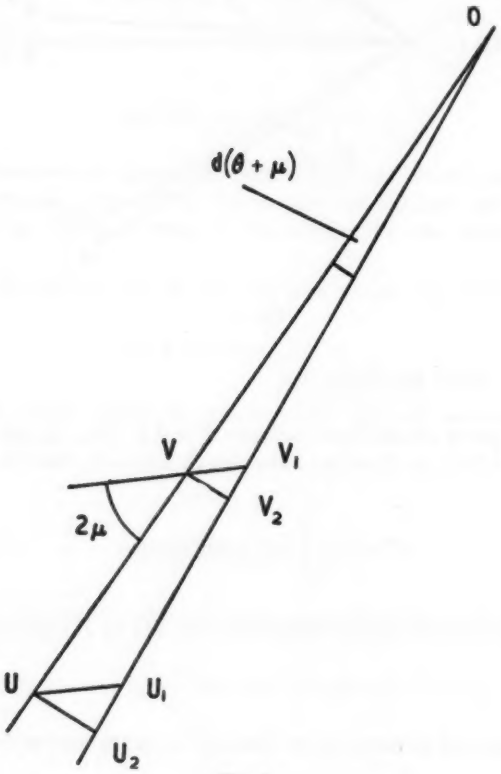

Fig. 3.

to the first order, and by (5), (3), and Bernoulli's equation

$$q^2(\gamma-1+2\sin^2\mu)=\text{constant},$$

$$\partial(\theta+\mu)/\partial\beta=\tfrac{1}{2}(1-d\mu/dt)=\tfrac{1}{4}(\gamma+1)\sec^2\mu=N. \quad . \quad . \quad (15)$$

Moreover, μ is constant along the straight Mach line UV, and hence, by (7),

$$(1/H_\beta)_V-(1/H_\beta)_U=[(h_\beta)_V-(h_\beta)_U]\sin 2\mu=-Nr, \quad . \quad . \quad (16)$$

where r denotes the distance from U to V. Differentiation of (16) gives (since $dr=h_\alpha d\alpha$)

$$\partial H_\beta/h_\alpha\partial\alpha=NH_\beta^2,$$

and hence, by (12), (5), and (15),

$$H_\beta\partial(\beta'/H_\beta)/h_\alpha\partial\alpha=\partial\beta'/h_\alpha\partial\alpha-NH_\beta\beta'=(1-N)H_\beta\alpha',$$

and upon integrating we have

$$(\beta'/H_\beta)_L-(\beta'/H_\beta)_G=(1-N)\int_G^L\alpha' dr,$$

i.e. by (16),

$$\beta'_L(1-NH_\beta{}^0r_L)=\beta'_G+(1-N)H_\beta{}^0\int_G^L\alpha' dr,$$

where $H_\beta{}^0$ is the value of H_β at G, and r_L the distance between G and L (repeated use is made of the fact that N is constant on a straight Mach line). The same equation holds with β' and α' replaced by β'' and α'', respectively. But $\beta''_G=\beta'_G$, since the correction does not alter the velocity components on the Mach line BA. Moreover, by (11) and (10), α'' has the same value at any point on GL as at the point on the axis lying on the same plus Mach line, and on the axis α'' is constant and equal to α'_A. Hence,

$$\beta'_L-\beta''_L=\frac{(1-N)H_\beta{}^0}{1-NH_\beta{}^0r_l}\int_G^L(\alpha'-\alpha'_A)\,dr. \quad . \quad . \quad . \quad (17)$$

It is convenient to extend the meaning of r so as to denote by it the distance, measured along the minus Mach lines, between the Mach line BA and an arbitrary

other plus Mach line in the region ABC (Fig. 1). Then, from Fig. 3,

$$V_1V_2/OV_2=U_1U_2/OU_2=\cot 2\mu\, d(\theta+\mu).$$

since μ is constant on a straight Mach line, and hence,

$$dr=U_1V_1-U_2V_2=V_1V_2-U_1U_2=-r\cot 2\mu\, d(\theta+\mu).$$

i.e., by (15), $$\partial r/\partial\beta=-Nr\cot 2\mu.$$

Moreover, $\alpha=$constant in the simple-wave region ABC, so

$$d\beta=-2dt=-2d\mu/(1-2N),$$

by (5) and (15), and hence,

$$r/r_0=\exp\int_{\mu_0}^{\mu}\frac{2N\cot 2\mu}{1-2N}\,d\mu=f(\mu)/f(\mu_0),$$

where $$f(\mu)=\{[(\gamma-\cos 2\mu)^{\gamma}(\sin\mu)^{-\gamma-1}]^{1/(\gamma-1)}\sec\mu\}^{\frac{1}{2}}. \quad . \quad . \quad (18)$$

r_0 may be interpreted as the value r takes on AC (Fig. 1) and by x^+ may be denoted the distance from A at which the axis is crossed by the plus Mach line intersecting AC at a distance r_0 from A (Fig. 1). Then

$$r_0\cos\mu_A=\tfrac{1}{2}x^+$$

and

$$r/f(\mu)=\tfrac{1}{2}x^+\sec\mu_A/f(\mu_A).$$

In particular

$$(x_L-x_G)/[f(\mu)\cos(\theta+\mu)]_G=\tfrac{1}{2}(x_Q-x_A)/[f(\mu)\cos\mu]_A, \quad . \quad . \quad (19)$$

and (17) may be written

$$\beta'_L-\beta''_L=\tfrac{1}{2}(f(\mu_A)\cos\mu_A)^{-1}\left[\frac{(1-N)H_\beta f(\mu)}{1-NH_\beta r_L}\right]_G\int_0^{x_Q-x_A}(\alpha'(x^+)-\alpha'_A)\,dx^+. \quad . \quad (20)$$

The correction (14) for the slope of the liner at any point L is seen to consist of two parts. The one, $\frac{1}{2}(\alpha'_Q-\alpha'_A)$, is due to the difference between the values of α on the plus Mach line through L in the uncorrected and the corrected nozzle,

respectively. The other is the contribution to the correction at L due to the total corrections applied to the liner upsream of L†.

The segment of the liner that is placed symmetrically to BC may be treated similarly. If l denotes the point with co-ordinates $x_l = x_L$, $y_l = -y_L$, then the correction to this liner segment is

$$y^* - y^m = \int_{x_B}^{x} \delta_l (1 + \tan^2 \theta_a)\, dx_L, \qquad (21)$$

and it is found that

$$\delta_l = -\tfrac{1}{2} (\alpha'_l - \alpha''_l + \beta'_Q - \beta'_A), \qquad (22)$$

and

$$\alpha'_l - \alpha''_l = \tfrac{1}{2} (f(\mu_A) \cos \mu_A)^{-1} \left[\frac{(1-N) H_\beta f(\mu)}{1 - N H_\beta r_L} \right]_a \int_0^{x_Q - x_A} (\beta'(x^+) - \beta'_A)\, dx^+ \qquad (23)$$

(use is made of the fact that $H_a(x, -y) = -H_\beta(x, y)$).

It may be noted that knowledge of α^m and β^m on the axis, in the test rhombus alone, is sufficient for the evaluation of the liner corrections (13) and (21). Such knowledge is not sufficient for the determination of, for example, the pressure on the liner segment BC of the uncorrected nozzle (for the Mach lines of the original nozzle design are also the characteristics of the hyperbolic system (8) and (9) and hence, the values of α' and β' on the axis in the test rhombus do not determine the values of these variables outside the test rhombus).

5. APPLICATION TO NOZZLE DESIGN

The most convenient procedure for the evaluation of the liner correction, for a nozzle giving a flow of uniform direction before correction, would appear to be as follows.

Consider the upper half of the nozzle (Fig. 1), and let A be the apex of the test rhombus, BA and AC the Mach lines through A in the field of flow of the original nozzle design, and BC the segment of the liner between them. To any point L on this liner segment there corresponds a point Q on the axis lying on the same plus

†If $(\beta'_L - \beta''_L)$ is neglected and x_Q is determined from x_L by the help of a drawing of the Mach line field, Harrop's[2] solution is obtained. If $(\beta'_L - \beta''_L)$ is neglected and L is taken to be the point of intersection of the straight line QR with the liner, Ward's[1] solution on Linear Theory is obtained.

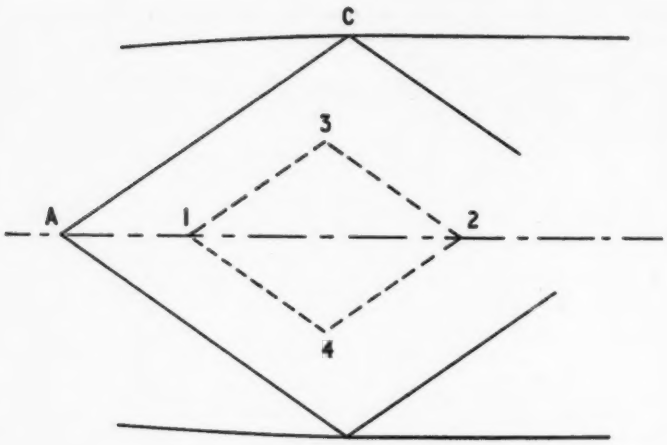

Fig. 4.

Mach line as L (Fig. 1), and a point G on the Mach line BA lying on the same minus Mach line as L.

The Mach line BA is the last Mach line that has to be computed point by point for the original nozzle design, and hence there is a set of points G on it at each of which the co-ordinates x_G, y_G, the Mach angle μ_G, the stream direction θ_G, the co-ordinates x_L, y_L of the corresponding point L, and the distance r_L between G and L have already been computed. For each of these points G find the abscissa x_Q of the corresponding point Q on the axis from (19) and (18).

If the flow in the test rhombus of the uncorrected nozzle is of uniform direction, $\theta' = \text{constant} = \theta'_A$ and (by (5)) $\alpha' = t' + \theta'_A$. Hence, by (4),

$$\alpha'_Q - \alpha'_A = (dt/dp)(p'_Q - p'_A) = -(\rho q^2 \tan\mu)^{-1}(p^m{}_Q - p^m{}_A) \quad . \quad . \quad (24)$$

at any point Q of the axis, where p^m is the pressure in the uncorrected nozzle, and ρ, q, μ are the density, velocity magnitude, and Mach angle, respectively in the test rhombus of the original nozzle design. Therefore, $(\beta'_L - \beta''_L)$ may be evaluated from (20) and (24) when the pressure distribution has been measured on the axis, in the test rhombus. In equation (20), x^+ denotes distance on the axis, measured downstream from A and N is defined by (15), $f(\mu)$ by (18); H_β must be found by numerical differentiation on the Mach line BA if the nozzle has been designed by the conventional method of characteristics†.

†$(H_\beta \sin 2\mu)^{-1} = h_\beta = (\partial x/\partial\beta)\sec(\theta-\mu)$, by (6) and (7) where $\partial x/\partial\beta$ is the ratio of the rates of change of x and β $(=(\pi/180)(\theta+P-1000°)$, where P is Busemann's pressure number) on the Mach line BA. Note that H_β vanishes like $\sqrt{(\beta-\beta_A)}$ near A, if the velocity gradient is continuous at A (Appendix of Ref. 5).

Finally, the correction δ_L to the inclination of the liner at L is found from (14), and the correction $(y^* - y^m)$ to the liner ordinate for the abscissa x from (13). The opposite correction must be applied to the liner in the lower half of the nozzle. The Mach number in the test rhombus of the corrected nozzle will be the same as that in the uncorrected nozzle at A.

6. TEST FOR UNIFORM STREAM DIRECTION

A check whether the stream direction in the test rhombus of the uncorrected nozzle is indeed uniform can be made by measuring the pressure at points in the test rhombus placed symmetrically with respect to the axis (Fig. 4). For if 1, 2, 3, 4 form a Mach quadrangle as shown, then by (5), (10) and (4),

$$\theta'_2 - \theta'_1 = t'_3 - t'_4 = -(\rho q^2 \tan \mu)^{-1} (p^m{}_3 - p^m{}_4), \quad . \quad . \quad . \quad (25)$$

and in the test rhombus θ' is equal to the actual stream direction in the uncorrected nozzle. (Only differences in θ' can be measured in this way.)

If the stream direction is not uniform, the distribution of θ' on the axis may be found by pressure measurements, except for an additive constant. For by the same argument it also follows that

$$\theta'_3 - \theta'_4 = -(\rho q^2 \tan \mu)^{-1} (p^m{}_2 - p^m{}_1), \quad . \quad . \quad . \quad . \quad (26)$$

and (22) and (23) hold for any four points in the test rhombus such that 1, 2 lie on a line parallel to the axis (with 2 downstream of 1) and 3, 4 complete the Mach quadrangle (with 1, 4, 2, 3 for the anti-clockwise order of the vertices, if the stream comes from the left).

When $\theta' - \theta'_A$ and $p^m - p^m{}_A$ have been measured on the axis, the corrections to the upper liner are found from (20), (14), (13) (and those to the lower liner from (23), (22), (21)) as described in Section 5, but instead of (24) the equations

$$\alpha'_Q - \alpha'_A = \theta'_Q - \theta'_A - (\rho q^2 \tan \mu)^{-1} (p^m{}_Q - p^m{}_A)$$

and

$$\beta'_Q - \beta'_A = \theta'_Q - \theta'_A + (\rho q^2 \tan \mu)^{-1} (p^m{}_Q - p^m{}_A)$$

which follow from (5) and (4) must be used.

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SOME DEVELOPMENTS OF EXPANSION METHODS FOR SOLVING THE FLUTTER EQUATIONS

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SUMMARY

In a theoretical flutter analysis, the real zeros $x<0$, $y>0$, of a stability determinant $\Delta(\sqrt{x}, y)$ which is a polynomial in y are required, and can readily be evaluated once the polynomial expansions are known for assigned values of x. When the air loads are estimated on the basis of classical derivative theory, Δ becomes a *bivariate* polynomial in $\sqrt{x}$ and y of a special type. Direct expansion of the stability determinant becomes too involved when the number of degrees of freedom is large and indirect methods of expansion are needed.

A method of bivariate expansion which uses a framework of lines subject to certain restrictions is examined in Part I, and is applied to a quaternary wing-flutter problem. The analysis shows that frameworks previously proposed, having all intersections outside the flutter quadrant ($x<0$, $y>0$), demand very high computational accuracy. A new framework is suggested having all intersections inside the flutter quadrant. The improvement in computational accuracy is considerable.

In Part II, methods for the univariate expansion of Δ with x assigned are briefly discussed, and some methods of bivariate expansion are given which provide useful alternatives to the framework method.

1. INTRODUCTION

In a theoretical flutter analysis, the aircraft structure is usually treated as a semi-rigid body with a finite number of degrees of freedom, n say, and the corresponding Lagrangian dynamical equations are expressed in non-dimensional form. It then follows that the flutter speeds and flutter frequencies of the system are given by the real zeros $x \equiv -\omega^2 < 0$, $y > 0$, of the n^{th} order stability determinant

$$\Delta(\sqrt{x}, y) = |ax + \mathscr{A}(\sqrt{x}) + ey| , \quad . \quad . \quad . \quad . \quad (1)$$

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where ω denotes the non-dimensional frequency parameter and y is a non-dimensional elastic stiffness—air speed parameter[1]. The elements of the matrices a and e are known real constants representing inertias and elastic stiffnesses respectively; structural dampings are neglected to avoid further complication of the problem. The elements of the matrix $\mathscr{A}(\sqrt{x})$ are in general real transcendental functions of $\sqrt{x}$ and represent the aerodynamic forces. Sometimes they are tabulated against ω, but in classical derivative theory they are expressed as quadratics in $\sqrt{x}$ as a simplifying approximation.

For convenience, let $x=s_x X$, $y=s_y Y$, where s_x and s_y are scale parameters which may be arbitrarily assigned. The stability determinant may be expressed with advantage in the form

$$\Delta(\pm\sqrt{x},y)=f(X,Y)\pm\sqrt{x}\,g(X,Y) \quad . \quad . \quad . \quad . \quad (2)$$

where $f(X,Y)$ and $g(X,Y)$ are real functions of X and Y, and are moreover polynomials in Y of degree n and $(n-1)$ respectively. The real zeros $x<0, y>0$ of $\Delta(\sqrt{x},y)$ are clearly the common zeros $X<0, Y>0$ of the functions $f(X,Y)$ and $g(X,Y)$. They can therefore be evaluated readily by examination of the polynomial expansions for $f(X,Y)$ and $g(X,Y)$ corresponding to assigned values of X, as illustrated in Appendix I. When classical derivative theory is employed, $f(X,Y)$ and $g(X,Y)$ become bivariate polynomials of degree n and $(n-1)$ respectively in X and Y jointly, and can therefore be specified completely by the values of $\Delta(\pm\sqrt{x},y)$ at only $\frac{1}{2}(n+1)(n+2)$ values of (x,y).

The problem of obtaining such bivariate expansions and univariate expansions (X assigned) is now examined in the case when the number of degrees of freedom is so large that direct expansion of $\Delta(\sqrt{x},y)$ becomes too involved.

Notation

A_{rs}, B_{rs}	framework bivariate expansion coefficients corresponding to functions $f(X,Y)$, $g(X,Y)$. *See* (5)
$\mathscr{A}(\sqrt{x})$	matrix of non-dimensional aerodynamic coefficients
a	matrix of non-dimensional inertial coefficients
a_r, b_r	polynomial expansion coefficients for f and g. See (11) and (17)
a_{rs}, b_{rs}	bivariate polynomial expansion coefficients for f and g. *See* (18)
C_k	Lagrangian expansion coefficients. *See* (14) and (22)
$D_1, \ldots, D_5$	special expansion coefficients for Frazer framework. *See* Section 3
e	matrix of non-dimensional elastic stiffness coefficients
f, g	functions related to expanded stability determinant. *See* (2)
f^+, f^-	sums of numerically positive and negative terms in expansion for f

f^{Δ}	f/f^{+}
g^{+}, g^{-}, g^{Δ}	defined in similar manner to f^{+}, f^{-}, f^{Δ}
L_r	line of framework. *See* (3)
n	number of degrees of freedom of flutter system
s_x, s_y	scale parameters for x and y
W_n	operator corresponding to Gregory-Newton formula. *See* (12), (13), (25) and (26)
X, Y	$x/s_x, y/s_y$
X_p, Y_t	co-ordinates of collocation points for expansion by Gregory-Newton formula. $X_p = X_0 + pv$, $Y_t = Y_0 + tw$
X_{rs}, Y_{rs}	co-ordinates of framework intersections
x	$-\omega^2$, where ω represents a non-dimensional frequency parameter
y	non-dimensional elastic stiffness—air speed parameter
Z	*See* (10)
Δ	stability determinant. *See* (1)
δ	operator denoting forward differences
ξ, η	$(X-X_0)/v$, $(Y-Y_0)/w$
Π	product sign
Σ	summation sign

PART I

A NEW FRAMEWORK OF LINES FOR THE FRAMEWORK METHOD OF BIVARIATE EXPANSION

2. THE FRAMEWORK METHOD OF BIVARIATE EXPANSION

This method for the bivariate expansion of the stability determinant corresponding to classical derivative theory was proposed by Frazer[2, 3]. Let a framework of $(n+2)$ lines

$$L_r \equiv Y - p_r X + q_r = 0, \ (r = 1 \text{ to } n+2) \ . \quad . \quad . \quad . \quad (3)$$

be chosen so that no two lines are parallel and no three are concurrent. Then the function $f(X, Y)$ may be expanded in the bivariate partial fractions form

$$f(X, Y)/\Pi L_k = \sum_{r \neq s} \{ A_{rs}/L_r L_s \} \quad . \quad . \quad . \quad . \quad (4)$$

or in the bivariate product form

$$f(X,Y)=\sum_{r\neq s}\{A_{rs}\prod_{k\neq r,\,s}L_k\}, \quad . \quad . \quad . \quad . \quad (5)$$

where k, r, and s range from 1 to $(n+2)$, and where the $\frac{1}{2}(n+1)(n+2)$ expansion coefficients A_{rs} are free for determination. If (X_{rs}, Y_{rs}) denotes the point of intersection of the lines $L_r=0$, $L_s=0$, it follows immediately that

$$A_{rs}=f(X_{rs},Y_{rs})/\prod_{k\neq r,\,s}L_k(X_{rs},Y_{rs}). \quad . \quad . \quad . \quad . \quad (6)$$

The formulæ relating to the function $g(X,Y)$ are similar, but only $(n+1)$ lines are needed and only $\frac{1}{2}n(n+1)$ expansion coefficients B_{rs} are involved. For the determination of a particular pair of expansion coefficients A_{rs} and B_{rs}, the evaluation of two real determinants or of a single complex determinant is required, according as X_{rs} is positive or negative.

The two frameworks proposed by Frazer[4, 3] and by Price[5], both having all intersections *outside* the flutter quadrant, are discussed in Section 3, and a new framework is suggested with all intersections inside the flutter quadrant (*i.e.* $X_{rs}<0, Y_{rs}>0$). Results obtained by the application of these frameworks to a quaternary wing-flutter problem are compared in Section 4.

One possible method for the determination of the required common zeros is by substitution of trial values for (X,Y) in the expansions (4) or (5), but speedier methods may be devised in which trial values are assigned only to one variable and the remaining variable is eliminated to give a test function. A method particularly suited for use with the new framework will be described later.

3. TYPES OF FRAMEWORK

If no framework lines cross the flutter quadrant, expansion in the bivariate partial fractions form (4) is most convenient. On the other hand, if some lines do cross the flutter quadrant, the product form (5) is best as large values occur with (4), and trial points can be usefully chosen to lie on these lines.

The Frazer framework

The framework proposed by Frazer[4] has the properties that all the intersections have positive abscissae, and that no lines cross the flutter quadrant. The lines are all members of the family

$$Y+0.1=0.25\,j(X-0.1\,j), \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where the values of j adopted for the framework are all positive integers and are chosen to yield well-separated intersections (X_{rs}, Y_{rs})*. Frazer usually omits the line L_{n+2} of the framework, in which case the expansions (4) and (5) require slight modification and only $(n+1)^2$ real determinants need be computed. The expansion (4) for $f(X, Y)$ is replaced by

$$f(X, Y)/\Pi L_k = \sum_{r \neq s} \{ A_{rs}/L_r L_s \} + \Sigma \{ D_r/L_r \}$$

where k, r and s range only from one to $(n+1)$.

The Price framework

The framework proposed by Price[5] is composed of lines all crossing the flutter quadrant, but having all intersections with positive abscissae. The lines are all tangents to the parabola $Y^2=4X$ at the parametric points $(s^2 k^2, 2sk)$ where k takes the integral values one to $(n+2)$ and s is a scale parameter which is to be arbitrarily assigned*. A useful transformation to new co-ordinates (ξ, η) may be employed, defined by the relations

$$X = s^2 \xi \eta, \quad Y = s(\xi + \eta). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

A new framework

It can readily be demonstrated that difficulties regarding accuracy are likely to be encountered with both the foregoing frameworks, but that they can be reduced by choosing a framework with some of its intersections in the neighbourhood of the zeros to be determined. If, in the product expansion (5) for $f(X, Y)$, a trial value of (X, Y) is chosen which is close to a zero $(\bar{X}, \bar{Y})$ then $f(X, Y) \longrightarrow 0$ as $(X, Y) \longrightarrow (\bar{X}, \bar{Y})$, but the separate terms in (5) will be non-zero and large in general. However, these terms could all vanish if one of the framework intersections (X_{rs}, Y_{rs}) happened also to lie at $(\bar{X}, \bar{Y})$. This suggests that if (X_{rs}, Y_{rs}) lay close to $(\bar{X}, \bar{Y})$, the terms of the summation would all be small and thus of an order of magnitude comparable with the value of $f(X, Y)$.

The new framework therefore has been chosen to have all its intersections in the flutter quadrant, $X<0$ and $Y>0$. The lines are all members of the family†

$$Y + 0.25\, j[X + 0.1(j-1)] = 0, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where the values of j adopted for the framework are all positive integers and are selected to yield well-separated intersections (X_{rs}, Y_{rs}). For the treatment of the quaternary problem in Section 4, the framework given by the values $j=0, 1, 2, 3, 5, 7$ is adopted (*see* Table I and Fig. 1.). A useful framework of *ten* lines is given by the values $j=0, 1, 2, 3, 5, 9, 13, 18, 23, 28$. In the product expansion for $g(X, Y)$,

*The scale parameters s_x and s_y both have the value unity.

†This family was obtained from the family used by Frazer by a simple linear transformation.

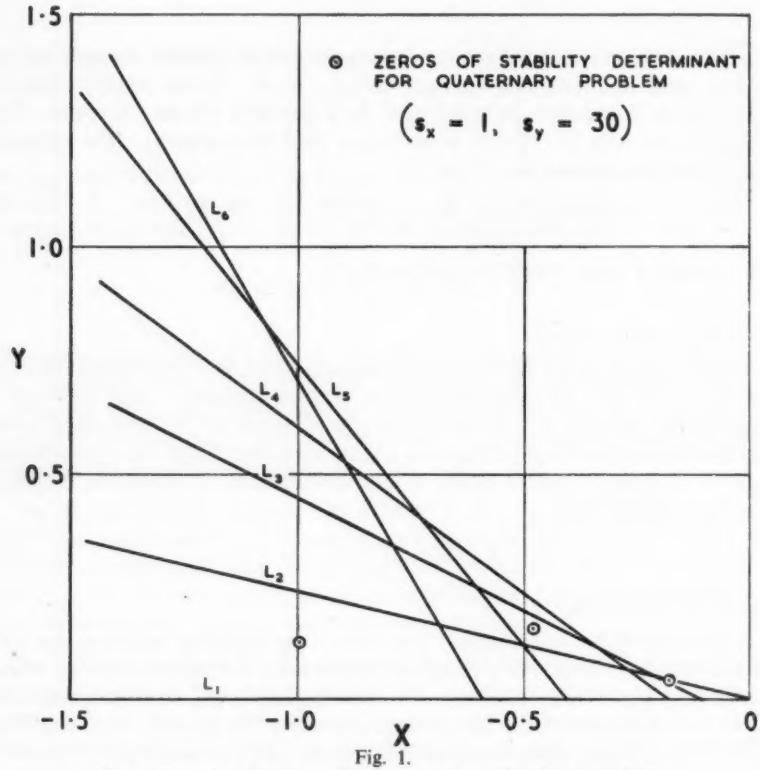

Fig. 1.
Lines of new framework used for quarternary wing-flutter problem.

where only $(n+1)$ of the $(n+2)$ lines used for $f(X, Y)$ are needed, the first line of the framework $(j=0)$ should be discarded. For the determination of a particular zero of $\Delta(\sqrt{x}, y)$ in a given problem, s_x and s_y should be so chosen that the zero is likely to occur in the region of the framework intersections.

In the treatment of the bivariate product expansions obtained for $f(X, Y)$ and $g(X, Y)$, a change of variables from (X, Y) to (Z, Y) may usefully be made, by means of the relation

$$Y+0.25\,Z\,[X+0.1\,(Z-1)]=0\,. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For, when Z is given one of the values adopted for j in the specification of the new framework, (X, Y) is effectively assigned to lie on a particular line of the framework, $L_s=0$ say; the summation in (5) is then with respect to r only and involves merely $(n+1)$ terms and expansion coefficients. Let some trial value be assigned to Z. The product expansions for $f(X, Y)$ and $g(X, Y)$ may then be arranged as polynomials in Y, whose coefficients are linear in the expansion coefficients A_{rs} and B_{rs} respectively. The multipliers of A_{rs} and B_{rs} in these combinations depend only on

TABLE I

New framework used for quaternary wing-flutter problem.

r	j	$-p_r$	q_r
1	0	0	0
2	1	0.25	0
3	2	0.50	0.05
4	3	0.75	0.15
5	5	1.25	0.50
6	7	1.75	1.05

$$L_r \equiv Y - p_r X + q_r$$
$$= Y + 0.25\, j\,[X + 0.1\,(j-1)]$$

r	s	X_{rs}	Y_{rs}
2	1	0	0
3	1	0.1	0
3	2	0.2	0.05
4	1	0.2	0
4	2	0.3	0.075
4	3	0.4	0.15
5	1	0.4	0
5	2	0.5	0.125
5	3	0.6	0.25
5	4	0.7	0.375
6	1	0.6	0
6	2	0.7	0.175
6	3	0.8	0.35
6	4	0.9	0.525
6	5	1.1	0.875

Z and the framework constants, and can be tabulated against Z if the framework is standardised. By a reduction scheme involving cross-multiplication between the coefficients of the polynomials, a test function $T(Z)$ can be evaluated which vanishes when the polynomials have a common real zero, and the corresponding value of Y can be simultaneously determined. A computational scheme for the case $n=4$ is set out in Appendix I. The relation (10) then gives the corresponding value of X.

4. NUMERICAL APPLICATIONS

A quaternary wing-flutter problem was treated in previous calculations with the Frazer framework[3]. In the present paper, the same problem is considered with the first six lines of the Price framework ($s=1.5$), and also with a framework consisting of the six lines of Table I, ($s_x=1$, $s_y=30$). Table II gives the expansion coefficients obtained with the three frameworks.

For a comparison of accuracy between the frameworks a *difference factor* will be introduced. At a chosen (X, Y) value, let f^+ and f^- denote the magnitudes of

TABLE II

Expansion coefficients for the quaternary wing-flutter problem

r	s	*The Frazer Framework*		*The Price Framework*		*The New Framework*	
		A_{rs}	B_{rs}	A_{rs}	B_{rs}	A_{rs}	B_{rs}
2	1	−0.01453933	−0.006318006	+0.01071499	+0.05811656	−0.087842194	
3	1	+0.3076831	+0.03942306	−0.4034104	−1.096037	+0.06243117	
3	2	−0.6523143	+0.4429171	+5.915443	+6.839501	+0.004803710	−0.005566485
4	1	−0.7530614	+0.3667179	+2.760917	+3.757726	−0.1895506	
4	2	+2.017487	−3.542989	−70.88806	−41.40342	−1.754462	+0.2301388
4	3	−0.7090571	+3.304325	+159.2768	+47.80535	−0.9891079	+0.9070491
5	1	+0.3910479	−1.112134	−5.851843	−3.190571	−0.6231269	
5	2	−0.4730186	+6.946636	+188.4149	+44.30491	+10.24894	−2.211498
5	3	−1.084819	−10.58620	−750.0547	−90.79215	+75.53664	−24.78022
5	4	+0.9792672	+4.148350	+556.0445	+34.18000	−102.1947	+82.96890
6	1			+3.731798		+1.322998	
6	2			−135.8661		−9.337961	+3.149724
6	3			+682.9612		−148.4662	+47.20418
6	4			−899.0205		+188.3665	−217.3752
6	5			+263.6495		+41.87863	+129.7036

$D_1 = +0.2268408$
$D_2 = -1.1185486$
$D_3 = +2.122015$
$D_4 = -1.383573$
$D_5 = +0.2202687$

the sums of the positive and negative terms respectively in the summation (5) for $f(X, Y)$; then the difference factor f is defined as

$$f^{\Delta} \equiv [\text{true value of } f(X, Y)]/f^{+}.$$

If, therefore, a rough estimate of f^{Δ} is made, it is possible to decide what accuracy is needed in the evaluation of f^{+} and f^{-}, and hence in the expansion coefficients and related products, to yield a specific accuracy in $f(X, Y)$. The symbols g^{+}, g^{-}, g^{Δ} have a similar significance in relation to $g(X, Y)$.

TABLE III

Relative accuracy of expansions given by various frameworks at some trial points for quaternary wing-flutter problem

x	y	f (*Frazer*)	f (*Price*)	f (*New*)	f^{Δ}(*Frazer*)	f^{Δ}(*Price*)	f^{Δ}(*New*)
−0.1125	1.425	+0.00001	–	+0.00000152	3×10^{-8}	5×10^{-11}	1×10^{-4}
−0.45	1.2	−0.00210	–	−0.00210138	5×10^{-5}	5×10^{-8}	1×10^{-1}
−0.225	2.925	+0.0001	–	+0.00006167	2×10^{-7}	5×10^{-10}	2×10^{-3}
−0.9	2.7	−0.0069	–	−0.0069839	1×10^{-5}	4×10^{-8}	1×10^{-2}
−0.405	4.41	–	–	−0.000188297	1×10^{-7}	6×10^{-10}	6×10^{-2}
−1.35	4.2	−0.003	–	−0.003796	1×10^{-6}	1×10^{-8}	7×10^{-4}
−0.4455	4.86	−0.001	–	−0.000960503	3×10^{-7}	3×10^{-9}	1×10^{-1}
−1.485	4.65	+0.002	–	+0.00168	5×10^{-7}	4×10^{-9}	2×10^{-4}
x	y	g (*Frazer*)	g (*Price*)	g (*New*)	g^{Δ}(*Frazer*)	g^{Δ}(*Price*)	g^{Δ}(*New*)
−0.1125	1.425	0.00031	–	0.0003150	4×10^{-6}	3×10^{-7}	2×10^{-3}
−0.45	1.2	0.0007	0.001	0.0006733	1×10^{-5}	6×10^{-7}	2×10^{-3}
−0.225	2.925	0.0025	–	0.0024899	5×10^{-6}	1×10^{-6}	1×10^{-2}
−0.9	2.7	0.0032	0.003	0.00325	6×10^{-6}	1×10^{-6}	6×10^{-4}
−0.405	4.41	0.002	0.002	0.00215397	1×10^{-6}	1×10^{-6}	1×10^{-1}
−1.35	4.2	0.007	0.006	0.00679	4×10^{-6}	2×10^{-6}	3×10^{-4}
−0.4455	4.86	0.003	0.003	0.00274923	1×10^{-6}	9×10^{-7}	1×10^{-1}
−1.485	4.65	0.008	0.008	0.00792	3×10^{-6}	2×10^{-6}	2×10^{-4}

Table III compares values of $f=f^+-f^-$, $g=g^+-g^-$, f^Δ, and g^Δ obtained with the various frameworks for some typical trial points. It is seen that the accuracy in the determination of $f(X, Y)$ and $g(X, Y)$ is much greater with the new framework than with the Frazer or Price frameworks. At trial points in the neighbourhood of the common zero $(x, y)=(-0.48, 4.86)$*, the determination of $f(X, Y)$ and $g(X, Y)$ to a particular accuracy requires the retention of at least five more significant figures with the Frazer framework and eight more with the Price framework than with the new framework.

5. CONCLUSIONS

With the new framework, the improvement in computational accuracy is considerable. Moreover, the computation required for the treatment of the bivariate product expansions can be reduced, since the framework lines cross the flutter quadrant and a simple change of variables may usefully be made. The construction of single-entry tables, for the general application of this framework to flutter problems, is desirable. Tables for the case $n=6$ are now in preparation and will be issued in due course.

PART II

OTHER EXPANSION METHODS

6. UNIVARIATE EXPANSION METHODS

Univariate expansion in Y is of particular interest in connection with flutter analysis at very low frequency parameters, when the approximations of classical derivative theory become inadequate.

When the elements of the aerodynamic matrix are given as tabulated functions of ω, Δ is polynomial in Y only. If $n\leqslant 4$, Δ may be expanded directly in Y for any assigned X, without much difficulty. For $n>4$, indirect methods of expansion are preferable. With X assigned, write

$$f(X, Y)=\sum_{r=0}^{n} a_r Y^{n-r}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Then the $(n+1)$ coefficients a_r can be calculated by collocation of the function $f(X, Y)$ at $(n+1)$ distinct points $Y=Y_t$, where t ranges from 0 to n.

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*This was the most important zero from the flutter standpoint. The other common zeros lay at $(x, y)=(-0.99, 3.57)$ and $(-0.17, 1.38)$.

For example, with *equally-spaced* collocation points $Y_t = Y_0 + tw$, the *Gregory-Newton interpolation formula* may be employed. With X assigned,

$$f(X, Y) = W_n(\eta) f(X, Y_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $W_n(\eta)$ denotes the differencing operator* defined by

$$W_n(\eta) = 1 + \eta\,\delta^1 + \frac{\eta(\eta-1)}{2!}\delta^2 + \ldots + \frac{\eta(\eta-1)(\eta-2)\ldots(\eta-n+1)}{n!}\delta^n. \quad (13)$$

Since $\eta = (Y - Y_0)/w$, the products in (13) can be easily expressed as polynomials in Y.

With collocation points *not necessarily equally spaced* (11) may be obtained in the *Lagrangian form*

$$f(X, Y) = \sum_{k=0}^{n} \{ C_k \prod_{t \neq k} (Y - Y_t) \} \quad . \quad . \quad . \quad . \quad (14)$$

where

$$C_k = f(X, Y_k) / \prod_{t \neq k} (Y_k - Y_t). \quad . \quad . \quad . \quad . \quad (15)$$

Alternatively, a method of *undetermined coefficients* may be applied, to provide a set of $(n+1)$ linear simultaneous equations

$$\sum_{r=0}^{n} a_r Y_t^{n-r} = f(X, Y_t) \quad . \quad . \quad . \quad . \quad . \quad (16)$$

for the coefficients a_r.

Expansion by the Gregory-Newton formula is the fastest of the preceding three methods. The formulæ appropriate to

$$g(X, Y) = \sum_{r=0}^{n-1} b_r Y^{n-r-1} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

are similar but only n collocation points Y_t are needed. Each of the methods

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*For convenience, the forward difference notation used here is

$$\delta^1 f(X, Y_t) = f(X, Y_{t+1}) - f(X, Y_t)$$

$$\delta^m f(X, Y_t) = \delta^{m-1} f(X, Y_{t+1}) - \delta^{m-1} f(X, Y_t).$$

described requires the evaluation of $(n+1)$ complex determinants $\Delta(\sqrt{X}, Y_t)$ for the determination of the polynomials $f(X, Y)$ and $g(X, Y)$ at any assigned value of $X<0$. From a computational standpoint, it would be helpful to standardise the set of collocation points to be employed, for the products in (13) and (14) could then be expressed as polynomials in Y, once for all, and the work required for the solution of the equations (16) could be reduced considerably.

More complicated methods which do not employ collocation can also be used, and these are possibly quicker. The stability determinant for an assigned x is first reduced to the simpler well-known form $|U(x)-yI|$ where $U(x)=e^{-1}[-ax-\mathscr{A}(\sqrt{x})]$, e^{-1} is the reciprocal of e, and I is the unit diagonal matrix. The modified determinant can then be expressed in y directly by methods reviewed and discussed by Hotelling[6] and Wayland[7].

7. BIVARIATE EXPANSION METHODS

When classical derivative theory is employed to give the aerodynamic forces, the function $f(X, Y)$ becomes a bivariate polynomial of degree n, jointly in X and Y. The coefficients in the Y-polynomial (11) are in fact polynomials of the form

$$a_r \equiv a_r(X) = \sum_{s=0}^{r} a_{rs} X^{r-s}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and in all there are $\frac{1}{2}(n+1)(n+2)$ disposable coefficients a_{rs}. The function $f(X, Y)$ may therefore be determined by collocation at $\frac{1}{2}(n+1)(n+2)$ distinct points (X_p, Y_t), where p ranges from 0 to n but t only from 0 to $(n-p)$.

Expansion based on univariate interpolation formulae

With $X=X_0$, the $(n+1)$ coefficients $a_r(X_0)$ in the Y-polynomial for $f(X_0, Y)$ can be deduced from a Gregory-Newton relation of the type (12) namely:—

$$\sum_{r=0}^{n} a_r(X_0) Y^{n-r} = W_n(\overline{Y-Y}_0/w)\, f(X_0, Y_0), \quad . \quad . \quad . \quad (19)$$

by comparison of the coefficients of the successive powers of Y in the left and right-hand sides. In particular $a_0(X)$ — which is independent of X — can be evaluated immediately as $a_0(X_0)$. Next, with $X=X_1$, the relation

$$\sum_{r=1}^{n} a_r(X_1) Y^{n-r} = W_{n-1}(\overline{Y-Y}_0/w)\,\{f(X_1, Y) - a_0(X_1) Y^n\}_{Y=Y_s} \quad . \quad (20)$$

gives the value of $a_r(X_1)$. Hence, $a_1(X)=a_{10}X+a_{11}$ is known at both X_0 and

X_1, and is therefore completely determinable. More generally, with $X=X_p$ at the $(p+1)^{th}$ stage the relation

$$\sum_{r=p}^{n} a_r(X_p)\,Y^{n-r} = W_{n-p}(\overline{Y-Y_0}/w)\,\{\,f(X_p,Y)-\sum_{r=0}^{p-1} a_r(X_p)\,Y^{n-r}\,\}_{Y=Y_*} . \qquad (21)$$

gives the values of $a_r(X_p)$ for $r=p$ to n^*. The polynomial $a_p(X)$ is now known at $(p+1)$ values of X and may therefore be determined completely by collocation in X at these points†. A simple computational scheme for the application of the foregoing method is outlined in Appendix II.

Treatments based on the Lagrangian form (14) or on undetermined coefficients can also be devised. At the $(p+1)^{th}$ stage the Lagrangian form provides the relation

$$\sum_{r=p}^{n} a_r(X_p)\,Y^{n-r} = \sum_{k}\{\,C_k \prod_{t\neq k}(Y-Y_t)\,\} \qquad . \quad . \quad . \qquad (22)$$

where k and t range from 0 to $(n-p)$, and where

$$C_k = [f(X_p,Y_k)-\sum_{r=0}^{p-1} a_r(X_p)\,Y_k^{n-r}]/\prod_{t\neq k}(Y_k-Y_t) \qquad . \quad . \qquad (23)$$

Alternatively, by the method of undetermined coefficients, the $(n-p+1)$ linear simultaneous equations ($t=0$ to $n-p$) are obtained

$$\sum_{r=p}^{n} a_r(X_p)\,Y_t^{n-r} = f(X_p,Y_t)-\sum_{r=0}^{p-1} a_r(X_p)\,Y_t^{n-r}, \qquad . \quad . \quad . \qquad (24)$$

for the coefficients $a_r(X_p)$.

Expansion by the Gregory-Newton formula is the fastest of the preceding three methods.

Expansion based on bivariate interpolation formulae

With the collocation points $X_p=X_0+pv$, $Y_t=Y_0+tw$ equally spaced in the X and Y variates separately, an extended form of the Gregory-Newton interpolation formula could be employed. It is easily shown[8] that

$$f(X,Y)=W_n(\xi,\eta)\,f(X_0,Y_0) \qquad . \quad . \quad . \quad . \qquad (25)$$

where $\xi=(X-X_0)/v$, $\eta=(Y-Y_0)/w$, and $W_n(\xi,\eta)$ represents the differencing

*Only $(n-p+1)$ collocation points $Y=Y_t$, $t=0$ to $(n-p)$, are employed at the $(p+1)^{th}$ stage.

†Alternatively, the value of $a_p(X)$ at any other value of X may be determined by interpolation.

operator* defined by

$$W_n(\xi,\eta)=1+\xi\,\delta^{1,0}+\eta\,\delta^{0,1}+\frac{\xi(\xi-1)}{2!}\delta^{2,0}+\xi\,\eta\,\delta^{1,1}+$$

$$\ldots+\frac{\eta(\eta-1)\ldots(\eta-n+1)}{n!}\delta^{0,n} \quad . \quad . \quad . \quad (26)$$

The products involving ξ and η can readily be expressed as bivariate polynomials in X and Y, and the bivariate differences can be built up simply and quickly by successive double-entry tabulations, beginning with the values for $f(X_p, Y_t)$.

General application

The formulae appropriate to $g(X, Y)$ are similar to those given for $f(X, Y)$ but only $\frac{1}{2}n(n+1)$ collocation points are needed. Each of the methods described requires the evaluation of $\frac{1}{2}(n+1)(n+2)$ complex determinants when all the collocation points have negative abscissae. It would be helpful to standardise the set of collocation points to be employed, for the products in the Gregory-Newton formulae and in the Lagrangian forms could then be expressed as polynomials, once for all. A useful set would be

$$X_p=-0.1\,p,\ Y_t=t, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

with the points (X_0, Y_t) discarded for the determination of $g(X, Y)$. If the values of the common zeros to be determined can be roughly estimated beforehand, the scale parameters s_x and s_y should be chosen so that some of the collocation points lie in their vicinity, as the computational accuracy will thereby be improved.

8. CONCLUSIONS

The bivariate expansion methods given in Part II involve more computational operations than the framework method, but the extra work is small compared with the amount needed for the evaluation of the determinants required in both cases.

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*This bivariate forward difference notation,

$$\delta^{1,0}f(X_p, Y_t)=f(X_{p+1}, Y_t)-f(X_p, Y_t),$$

$$\delta^{0,1}f(X_p, Y_t)=f(X_p, Y_{t+1})-f(X_p, Y_t),$$

$$\delta^{l,m}f(X_p, Y_t)=\delta^{l-1,m}f(X_{p+1}, Y_t)-\delta^{l-1,m}f(X_p, Y_t),$$

$$=\delta^{l,m-1}f(X_p, Y_{t+1})-\delta^{l,m-1}f(X_p, Y_t),$$

is a natural extension of the difference notation used earlier.

Some improvement in accuracy compared with that of the framework method is likely, as the collocation points can be distributed *evenly* over the region in which the required zeros are expected to occur, whereas the line intersections of a framework are of necessity unevenly spaced.

APPENDIX I

Cross-multiplication treatment of polynomial expansions ($n=4$)

When the polynomial expansions

$$f(X,Y)=\sum_{r=0}^{n} a_r Y^{n-r},\quad g(X,Y)=\sum_{r=0}^{n-1} b_r Y^{n-r-1},$$

have a common zero at the assigned X (or Z), the Y-eliminant J ($\equiv f_0 g_1 - f_1 g_0$) derived later vanishes, and the corresponding Y-value is given by either of the two identical expressions $-f_1/f_0$, $-g_1/g_0$. For the determination of the common zeros, it is more accurate to examine the test function $T=J/(f_0 g_1)$ rather than J. Alternatively, the values of Y given by the above two expressions could be compared.

$r=0$	1	2	3	4	*Operation*	*Checks**	
						Sum of coeffs.	*Cross-multiplied sums*
a_0	a_1	a_2	a_3	a_4		$\Sigma\, a_r$	
b_0	b_1	b_2	b_3			$\Sigma\, b_r$	
c_0	c_1	c_2	c_3		$a_0\, b_r - b_0\, a_r$	$\Sigma\, c_r$	$a_0 \Sigma\, b_r - b_0 \Sigma\, a_r$
d_0	d_1	d_2			$b_0\, c_r - c_0\, b_r$	$\Sigma\, d_r$	$b_0 \Sigma\, c_r - c_0 \Sigma\, b_r$
e_0	e_1	e_2			$c_0\, d_r - d_0\, c_r$	$\Sigma\, e_r$	$c_0 \Sigma\, d_r - d_0 \Sigma\, c_r$
f_0	f_1				$d_0\, e_r - e_0\, d_r$	$\Sigma\, f_r$	$d_0 \Sigma\, e_r - e_0 \Sigma\, d_r$
g_0	g_1				$e_0\, f_r - f_0\, e_r$	$\Sigma\, g_r$	$e_0 \Sigma\, f_r - f_0 \Sigma\, e_r$
J					$f_0\, g_1 - g_0\, f_1$		

*Checks on the numerical working are provided by opposite entries in these two columns.

APPENDIX II

Computational scheme for bivariate expansion method based on univariate Gregory-Newton interpolation formula

The scheme to be outlined here involves essentially the formation of simple difference tables and some inverse interpolation. The values of $f(X_p, Y_t)$, where p ranges from 0 to n but t only from 0 to $(n-p)$, are assumed known at the outset.

For the first stage with $X=X_0$, the values $f(X_0, Y_t)$ are listed against Y_t, $t=0$ to n, and the corresponding differences are formed. The coefficients $a_r(X_0)$ for $r=0$ to n then follow immediately from the relation (19)†. It is convenient next to prepare to tabulate each coefficient $a_r(X)$ at the values $X=X_p$, $p=0$ to n, and to make difference tables as illustrated below. The column $a_0(X)$ can be completed immediately, since $a_0(X)=\text{constant}=a_0(X_0)$.

	$a_0(X)$	$a_1(X), \delta^1 a_1(X)$	. . . .	$a_p(X), \delta^1 a_p(X), \ldots, \delta^p a_p(X)$	. . . .
X_0	*	*		*	
		*		*	
		*		*	
				.	
				.	*
X_p	constant	constant		.	
				*	
				*	constant
X_n					

The values $f(X_1, Y_t) - a_0(X_1) Y_t^n$, $t=0$ to $(n-1)$, can now be listed against Y_t and the differences evaluated, so that the values of $a_r(X_1)$ for $r=1$ to n follow immediately from the relation (20). Now the first order differences are constant since $a_1(X)$ is linear in X, so from the two known values $a_1(X_0)$ and $a_1(X_1)$ the column $a_1(X)$ may be completed by inverse interpolation.

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†It is desirable that a standard set of collocation points should be employed for which the expressions $[\eta(\eta-1)\ldots(\eta-m+1)]/m!$ have been expressed once for all as polynomials in Y.

Similarly, with $X = X_p$ at the $(p+1)^{th}$ stage, the values of

$$f(X_p, Y_t) - \sum_{r=0}^{p-1} a_r(X_p)\, Y_t^{n-r},\ t=0 \text{ to } (n-p),$$

may be listed against Y_t and the differences evaluated, so that the values of $a_r(X_p)$ for $r=p$ to n follow immediately from the relation (21). But the p^{th} order differences are constant, since $a_p(X)$ is a polynomial of order p in X, so from the known values $a_p(X_0)$, $a_p(X_1)$, ..., $a_p(X_p)$, the column $a_p(X)$ may be completed by inverse interpolation.

Thus on completion of the $(n+1)^{th}$ stage with $X = X_n$, there is to hand a complete table of each coefficient $a_r(X)$ at the values $X = X_p$, $p=0$ to n, together with the corresponding differences. Hence, the value of each coefficient $a_r(X)$ at any intermediate X may be found by interpolation, or the corresponding polynomial in X may be determined directly, by means of the Gregory-Newton formula.

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ERRATA

ERRATUM—Volume II, Part II, August 1950.

The following correction should be made in Volume II, Part II of the August 1950 number to "Critical Mach Numbers for Swept-Back Wings," by S. Neumark, equation (18), page 93:

$$\text{For} \left(\frac{1-\xi}{1+\xi} \frac{1-\sin\varphi}{1+\sin\varphi} \right) \text{read} \left(\frac{1+\xi}{1-\xi} \frac{1+\sin\varphi}{1-\sin\varphi} \right).$$